A Portrait of Linear Algebra

Fourth Edition

Selected Answers to the Exercises

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Chapter Zero Exercises

1. Answers:
   a. a true logical statement.
   b. a logical statement, but it is False, because $-5 < 3$ but $25 > 9$.
   c. a true logical statement.
   d. a false logical statement, because if $x < 0$, then $\sqrt{x}$ is imaginary.
   e. a true logical statement as of March 2020, with 237 consecutive weeks.
   f. not a logical statement, because it cannot be ascertained to be True or False ("best" is not a well-defined adjective; unlike the previous Exercise, where "most number of consecutive weeks as number 1" is well defined).
2. Answers:
   a. Converse: If $\cos(x) \geq 0$, then $0 \leq x \leq \pi/2$. Inverse: If $x > \pi/2$ or $x < 0$, then $\cos(x) < 0$. Contrapositive: If $\cos(x) < 0$, then $x > \pi/2$ or $x < 0$.
   b. Converse: If $f(x)$ possesses both a maximum and a minimum on $[a,b]$, then $f(x)$ is continuous on $[a,b]$. Inverse: If $f(x)$ is not continuous on $[a,b]$, then $f(x)$ either does not possess an absolute maximum or an absolute minimum on $[a,b]$. Contrapositive: If $f(x)$ does not possess either an absolute maximum or an absolute minimum on $[a,b]$, then $f(x)$ is not continuous at $x = a$.
3. Answers:
   a. $A \cup B = \{a,b,c,f,g,h,i,j,m,p,q\}$, $A \cap B = \{c,h,j\}$,
   $A - B = \{a,f,i,m\}$, $B - A = \{b,g,p,q\}$.
   b. $A \cup B = \{a,b,d,g,h,j,k,p,q,r,s,t,v\}$, $A \cap B = \{d,g,h,p,t\}$,
   $A - B = \{a,j,r\}$, $B - A = \{b,k,q,s,v\}$.
6. (b) "If n does not have a prime factor which is at most $\sqrt{n}$, then n is prime." (c) The number 11303 is composite. One prime factor is smaller than 100.
8. Answers:
   a. There exists a real number x which does not have a multiplicative inverse. The negation is true, and the original statement is false.
   b. For all real numbers x: $x^2 > 0$. The negation is true, and the original statement is false.
   c. For all negative numbers x: $x^2 \neq 4$. The original statement is true, and the negation is false.
   d. There exists a prime number which is even. The negation is true (2 is an even prime number), and the original statement is false.
10. 2027 and 2029.
11. 233
14. (e) For any two sets X and Y : $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$.
16. (a) $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$; 8 subsets.
    (c) you get exactly the same list as the subsets on the right column.

Chapter One Exercises

1.1 Exercises

1. (b) $\|\vec{u}\| = \sqrt{65}$; (c) $\vec{u}_1 = \frac{1}{\sqrt{65}}\langle -4,7\rangle$ and $\vec{u}_2 = \frac{-1}{\sqrt{65}}\langle -4,7\rangle$ (d) $3\vec{v} = \langle 9,15\rangle$, $5\vec{w} = \langle 5,-10\rangle$, $\vec{v}+5\vec{w} = \langle 14,5\rangle$ and $3\vec{v}-5\vec{w} = \langle 4,25\rangle$
2. (b) $2\vec{u} = \langle 10,-6,4\rangle$, $3\vec{w} = \langle -6,15,12\rangle$, $2\vec{u}+3\vec{w} = \langle 4,9,16\rangle$ and $2\vec{u}-3\vec{w} = \langle 16,-21,-8\rangle$
   (c) $\|\vec{w}\| = \sqrt{45} = 3\sqrt{5}$ (d) $\vec{u}_1 = \frac{1}{3\sqrt{5}}\langle -2,5,4\rangle$ and $\vec{u}_1 = \frac{-1}{3\sqrt{5}}\langle -2,5,4\rangle$.
   (e) i. $-\frac{3}{5}\vec{w} = \langle 6/5,-3,-12/5\rangle$ ii. $2\vec{u}+5\vec{v} = \langle 30,-6,-31\rangle$
   iii. $3\vec{w}-4\vec{u} = \langle -26,27,4\rangle$ iv. $-4\vec{u}+7\vec{v}-2\vec{w} = \langle 12,2,-65\rangle$.
3. (a) $\vec{u}+\vec{v} = \langle 1,-2,7,3\rangle$ (b) $\vec{u}+\vec{w} = \langle -1,-3,4,-2\rangle$ (c) $\vec{v}-\vec{w} = \langle 2,1,3,5\rangle$
   (d) $-2\vec{u} = \langle -6,10,-2,-14\rangle$ (e) $\frac{3}{4}\vec{v} = \left\langle -\frac{3}{2},\frac{9}{4},\frac{9}{2},-3\right\rangle$
   (f) $-\frac{5}{3}\vec{w} = \left\langle \frac{20}{3},-\frac{10}{3},-5,15\right\rangle$
   (g) $5\vec{u}+3\vec{v} = \langle 9,-16,23,23\rangle$ (h). $-\frac{3}{2}\vec{u}+\frac{5}{4}\vec{v} = \left\langle -7,\frac{45}{4},6,-\frac{31}{2}\right\rangle$
   (i) $2\vec{u}-3\vec{v}+7\vec{w} = \langle -16,-5,5,-37\rangle$ (j) $-5\vec{u}+2\vec{v}-4\vec{w} = \langle -3,23,-5,-7\rangle$
   (k) $-\frac{3}{2}\vec{u}+\frac{3}{4}\vec{v}-\frac{5}{3}\vec{w} = \left\langle \frac{2}{3},\frac{77}{12},-2,\frac{3}{2}\right\rangle$ (l) $\frac{3}{2}\vec{u}-\frac{3}{4}\vec{v}+2\vec{w} = \left\langle -2,-\frac{23}{4},3,-\frac{9}{2}\right\rangle$
4. Answers:
   a. $\vec{u} = \langle -15,6,7\rangle$ and $\vec{v} = \langle 42,-17,-16\rangle$.
   b. Yes: $\langle -3,7\rangle = 40\langle 5,-2\rangle + 29\langle -7,3\rangle$.
   c. Yes: $\langle -17,-9,29,-37\rangle = 5\langle 3,-5,1,7\rangle + 8\langle -4,2,3,-9\rangle$.
   d. No: Using the first two coordinates, we get $x = -4$ and $y = 9$, but although these satisfy the 3rd coordinate, they do not satisfy the 4th.
   e. $\vec{u} = \langle -3,4,2,6,-7\rangle$ and $\vec{v} = \langle -1,-3,5,-3,2\rangle$
   f. $(7,-3)$
   g. $(-4,1,7)$
   h. $\vec{u} = \langle -4,4,-8\rangle$
6. (d) Contrapositive: if $\vec{u} = \langle u_1,u_2\rangle$ and $\vec{v} = \langle v_1,v_2\rangle$ are vectors in $\mathbb{R}^2$, then they are ***not parallel*** to each other ***if and only if*** $u_1v_2 - u_2v_1 \neq 0$.
8. PQ is 26 cm. long.

1.2 Exercises

1. $y = 4x/7$
2. $y = -5x/3$
3. $x = 5t,\ y = -4t,\ z = 2t$, and $t = x/5 = y/(-4) = z/2$
4. $x = -t,\ y = 3t,\ z = -6t$, and $t = -x = y/3 = z/(-6)$
5. Not possible because the direction vector has 0 in the y-component.
6. $2x - 11y + z = 0$
7. $31x - 29y - 13z = 0$
8. $10x - 2y + 15z = 0$. We must solve for s from y, solve for r from z, then substitute these into x.
9. It is a line, because the two vectors are parallel.
14. d. The line is not on the plane.

    e. The line is on the plane.

1.3 Exercises

1. $\langle x, y, z\rangle = \langle 2 - 3t, -7 + 6t, 4 + 8t\rangle$, and

   $t = \dfrac{x-2}{-3} = \dfrac{y+7}{6} = \dfrac{z-4}{8}$
2. $\langle x, y, z\rangle = \langle 3 + 2t, 2, -5 - 5t\rangle$, and

   $x = 3 + 2t,\ y = 2,\ z = -5 - 5t$.

   It is not possible because the direction vector has 0 in the y-component.
3. $\vec{v} = \overrightarrow{PQ} = \langle 4, -2, 3\rangle$, so $\langle x, y, z\rangle = \langle -4 + 4t, 3 - 2t, -5 + 3t\rangle$ is one possible answer (other answers are possible).

   $t = \dfrac{x+4}{4} = \dfrac{y-3}{-2} = \dfrac{z+5}{3}$
4. $\vec{v} = \overrightarrow{PQ} = \langle 3, -4, 0\rangle$, so $\langle x, y, z\rangle = \langle 5 + 3t, 3 - 4t, -2\rangle$ is one possible answer (other answers are possible).

   It is not possible because the direction vector has 0 in the z-component.
5. $x - 3y + z = -1$
6. $9x + 10y - 2z = 28$
7. They determine a line because the vector $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$.
10. $21x + 13y + z = 114$
11. $7x + 2y + 4z = 15$
12. $\left(-\dfrac{65}{29}, \dfrac{21}{29}, \dfrac{52}{29}\right)$
15. b. $\dfrac{x-5}{3} = \dfrac{y+2}{5} = -z + 4$

    c. $\langle x, y, z\rangle = \langle -1 + 3t, -7 + 5t, 3 - t\rangle$

16. $13x - 7y + 4z = 95$
17. $17x - 4y + 22z = -80$
20. $y = -\frac{7}{2}x + \frac{29}{2}$; $(3,4)$ is another point, but there are an infinite number of other answers.
25. The critical value is $t = \frac{-3}{\sqrt{66}}$; $\left(\frac{53}{11}, -\frac{67}{22}, \frac{51}{22}\right)$; distance: $\frac{7}{22}\sqrt{374}$
26. The critical value is $t = \frac{14}{\sqrt{30}}$; $\left(\frac{98}{15}, \frac{7}{3}, -\frac{46}{15}\right)$; distance: $\frac{1}{15}\sqrt{25530}$
27. The distance is $\frac{\sqrt{1235}}{\sqrt{14}}$ or $\frac{\sqrt{17290}}{14}$
28. The distance is $\frac{12}{5}$ or 2.4

1.4 Exercises

1. Answers:
   a. valid, Type 1.
   b. valid, Type 3.
   c. not a row operation.
   d. valid, Type 1.
   e. valid, Type 2.
   f. Replace row 5 with row 2.
   g. valid, Type 3.
   h. not valid: these are two elementary row operations.
   i. not a row operation.
   j. not valid: these are two elementary row operations.
   k. valid, Type 3.
   l. not valid: these are three elementary row operations.
2. Answers:
   a. $\langle x_1, x_2, x_3 \rangle = \langle -3, 2, 6 \rangle$
   b. $\langle x_1, x_2, x_3 \rangle = \langle 7, 0, -4 \rangle$
   c. $\langle x_1, x_2, x_3 \rangle = \langle -3 - 7r, 2 + 4r, r \rangle$
   d. $\langle x_1, x_2, x_3 \rangle = \langle 6 + 3r, r, -7 \rangle$
   e. $\langle x_1, x_2, x_3 \rangle = \langle 8 + 5r - 2s, r, s \rangle$
   f. $\langle x_1, x_2, x_3, x_4 \rangle = \langle 5 - 3r, 6 + 2r, r, -4 \rangle$
   g. $\langle x_1, x_2, x_3, x_4 \rangle = \langle 3 + 5r, -4r, -2 + 7r, r \rangle$
   h. no solutions
   i. $\langle x_1, x_2, x_3, x_4 \rangle = \langle -2 + 5r, r, 3, 7 \rangle$
   j. $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle -5 - 7r - 5s, 2 + 4r + 3s, 4 - 6r + 2s, r, s \rangle$
   k. $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle -5 - 6r, 2 + 3r, 4 - 2r, -1 - 8r, r \rangle$

l. $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle -2 + 5r - 4s, r, 9 - 7s, 6 - 3s, s \rangle$
m. $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle 5 - 3r + 4s + 6t, -1 + 2r + 9s - 8t, r, s, t \rangle$
n. $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle r, 2 + 3s, s, -7, 4 \rangle$
o. $\langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle = \langle 7 + 9r - 4s, -3r + s, r, -1 - 6s, 2 - 5s, s \rangle$
p. $\langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle = \langle 4 + 5r - 3s, 5 - 3r, -2 + 2r - 4s, 3 - 7r + 6s, r, s \rangle$

3. Answers:
a. $\langle -3, 2, 0 \rangle$; $\langle -17, 10, 2 \rangle$
b. $\langle 6, 0, -7 \rangle$; $\langle 3, -1, -7 \rangle$
c. $\langle 8, 0, 0 \rangle$; $\langle 47, 5, -7 \rangle$
d. $\langle 5, 6, 0, -4 \rangle$; $\langle 11, 2, -2, -4 \rangle$
e. $\langle 3, 0, -2, 0 \rangle$; $\langle 18, -12, 19, 3 \rangle$
f. $\langle -2, 0, 3, 7 \rangle$; $\langle -27, -5, 3, 7 \rangle$
g. $\langle -5, 2, 4, -1, 0 \rangle$; $\langle -35, 17, -6, -41, 5 \rangle$
h. $\langle -2, 0, 9, 6, 0 \rangle$; part (ii) has no solution.
i. $\langle -5, 2, 4, 0, 0 \rangle$; $\langle -3, 1, 12, -1, 1 \rangle$
j. $\langle 0, 2, 0, -7, 4 \rangle$; $\langle 4, 14, 4, -7, 4 \rangle$
k. $\langle 7, 0, 0, -1, 2, 0 \rangle$; $\langle 0, 5/2, -1, 2, 9/2, -1/2 \rangle$
l. $\langle 4, 5, -2, 3, 0, 0 \rangle$; $\langle 9, -5, -98/9, 3, 10/3, 35/9 \rangle$

1.5 Exercises

1. ***Assisted computation:***
a. $\vec{x} = \langle 7 - 5x_3, -4 + 2x_3, x_3 \rangle$, x_3 free; $\vec{b} = 7\vec{v}_1 - 4\vec{v}_2$
b. $\vec{x} = \langle 4 - 5x_4, -2 + 3x_4, -1 + 2x_4, x_4 \rangle$, x_4 free; $\vec{b} = 4\vec{v}_1 - 2\vec{v}_2 - \vec{v}_3$
c. $\vec{x} = \langle 5 - 7x_3, 3 + 4x_3, x_3, -7 \rangle$, x_3 free; $\vec{b} = 5\vec{v}_1 + 3\vec{v}_2 - 7\vec{v}_4$
d. $\vec{x} = \langle -7 - 4x_3 - 9x_4, 6 + 2x_3 + 5x_4, x_3, x_4 \rangle$, x_3, x_4 free; $\vec{b} = -7\vec{v}_1 + 6\vec{v}_2$
e. $\vec{x} = \langle -3 + 3x_2 - 2x_4, x_2, 2 + x_4, x_4 \rangle$, x_2, x_4 free; $\vec{b} = -3\vec{v}_1 + 2\vec{v}_3$
f. no solutions
g. $\vec{x} = \langle 4, 7, -3 \rangle$; $\vec{b} = 4\vec{v}_1 + 7\vec{v}_2 - 3\vec{v}_3$
h. $\vec{x} = \langle 2 - 2x_3, 5 - 3x_3, x_3 \rangle$, x_3 free; $\vec{b} = 2\vec{v}_1 + 5\vec{v}_2$
i. $\vec{x} = \langle 4 - 3x_4, -3 + 5x_4, -5 + 4x_4, x_4 \rangle$, x_4 free; $\vec{b} = 4\vec{v}_1 - 3\vec{v}_2 - 5\vec{v}_3$
j. $\vec{x} = \langle 3 - 2x_5, -5 - 6x_5, 4 + 3x_5, -6 - 4x_5, x_5 \rangle$, x_5 free; $\vec{b} = 3\vec{v}_1 - 5\vec{v}_2 + 4\vec{v}_3 - 6\vec{v}_4$
k. $\vec{x} = \langle 2 - 7x_3, 6 - 5x_3, x_3, -3, 4 \rangle$, x_3 free; $\vec{b} = 2\vec{v}_1 + 6\vec{v}_2 - 3\vec{v}_4 + 4\vec{v}_5$
l. $\vec{x} = \langle 3 - 5x_4 - 6x_5, 2 + 3x_4 + 8x_5, -6 - 7x_4 - 10x_5, x_4, x_5, -9 \rangle$, x_4, x_5 free;
$\vec{b} = 3\vec{v}_1 + 2\vec{v}_2 - 6\vec{v}_3 - 9\vec{v}_6$
m. $\vec{x} = \langle -3 - 4x_3 - 6x_5, 5 - 3x_3 - 4x_5, x_3, -3x_5, x_5, 7 \rangle$, x_3, x_5 free;
$\vec{b} = -3\vec{v}_1 + 5\vec{v}_2 + 7\vec{v}_6$
n. $\vec{x} = \langle 3 - 7x_3 - 2x_6, -5 - 5x_3 - 6x_6, x_3, 4 + 3x_6, -6 - 4x_6, x_6 \rangle$, x_3, x_6 free;

$\vec{b} = 3\vec{v}_1 - 5\vec{v}_2 + 4\vec{v}_4 - 6\vec{v}_5$

2. ***Particular Solutions***:
   a. $\vec{x} = \langle -18, 6, 5 \rangle$
   b. $\vec{x} = \langle 19, -11, -7, -3 \rangle$
   c. $\vec{x} = \langle -9, 11, 2, -7 \rangle$
   d. $\vec{x} = \left\langle \frac{29}{20}, \frac{29}{20}, -\frac{13}{20}, -\frac{13}{20} \right\rangle$
   e. $\vec{x} = \langle 3, 2, -7, 2 \rangle$
   f. $\vec{x} = \left\langle -\frac{3}{4}, \frac{3}{4}, 4, 2 \right\rangle$
   g. $\vec{x} = \left\langle \frac{2}{3}, 3, \frac{2}{3} \right\rangle$
   h. $\vec{x} = \left\langle -\frac{7}{5}, 6, \frac{11}{5}, \frac{9}{5} \right\rangle$
   i. no solution (not possible with the given condition).
   j. $\vec{x} = \left\langle -\frac{146}{25}, \frac{2}{5}, \frac{28}{25}, -3, 4 \right\rangle$
   k. $\vec{x} = \left\langle \frac{47}{10}, \frac{751}{30}, -\frac{371}{30}, -\frac{69}{10}, \frac{82}{15}, -9 \right\rangle$
   l. $\vec{x} = \left\langle -\frac{17}{4}, \frac{39}{8}, -\frac{17}{8}, -\frac{39}{8}, \frac{13}{8}, 7 \right\rangle$
3. Answers:
   a. $\vec{x} = \langle -3 + 5z, 7 - 4z, z \rangle$, z is free.
   b. $\vec{x} = \langle 5, -3, -2 \rangle$.
   c. $\vec{x} = \langle -4 - 5w, 2 + 3w, 1 + 2w, w \rangle$, w is free.
   d. $\vec{x} = \langle 8 - 4z - 7w, -6 + 3z + 5w, z, w \rangle$, z and w are free.
   e. $\vec{x} = \langle 8 - 7z, 5 + 4z, z, -10 \rangle$, z is free.
   f. $\vec{x} = \langle 7 - 7x_3 - 5x_5, 2 + 4x_3 - 3x_5, x_3, -6 + 7x_5, x_5 \rangle$, x_3 and x_5 are free.
4. ***Membership in a Span***:
   a. (i) yes; (ii) $\vec{x} = \langle 3, 5 \rangle$; (iii) $\vec{b} = 3\vec{v}_1 + 5\vec{v}_2$
   b. (i) no solutions
   c. (i) yes; (ii) $\vec{x} = \langle 5, -2, 4 \rangle$; (iii) $\vec{b} = 5\vec{v}_1 - 2\vec{v}_2 + 4\vec{v}_3$
   d. (i) yes; (ii) $\vec{x} = \langle 5 - 3x_3, 7 - 4x_3, x_3 \rangle$; (iii) $\vec{b} = 5\vec{v}_1 + 7\vec{v}_2$
   e. (i) yes; (ii) $\vec{x} = \langle 6 - 5x_4, -4 + 3x_4, 5 - 4x_4, x_4 \rangle$; (iii) $\vec{b} = 6\vec{v}_1 - 4\vec{v}_2 + 5\vec{v}_3$
   f. (i) yes; (ii) $\vec{x} = \langle 4, -2, -5 \rangle$; (iii) $\vec{b} = 4\vec{v}_1 - 2\vec{v}_2 - 5\vec{v}_3$
   g. (i) yes; (ii) $\vec{x} = \langle 2 + x_3, -5 - 2x_3, x_3 \rangle$; (iii) $\vec{b} = 2\vec{v}_1 - 5\vec{v}_2$
   h. (i) yes; (ii) $\vec{x} = \langle 4 - x_3, -3 + 2x_3, x_3, -6 \rangle$; (iii) $\vec{b} = 4\vec{v}_1 - 3\vec{v}_2 - 6\vec{v}_4$
5. ***More on Particular Solutions***:
   a. $\vec{b} = -\frac{1}{4}\vec{v}_1 + \frac{7}{4}\vec{v}_2$
   b. $\vec{b} = \frac{1}{3}\vec{v}_2 + \frac{5}{3}\vec{v}_3$

c. $\vec{b} = -\frac{2}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_3 + \frac{4}{3}\vec{v}_4$

d. $\vec{b} = -\frac{1}{4}\vec{v}_1 - \frac{1}{4}\vec{v}_2 + \frac{5}{4}\vec{v}_4$

e. not possible

f. $\vec{b} = \frac{5}{4}\vec{v}_1 + \frac{5}{2}\vec{v}_2 + \frac{11}{4}\vec{v}_3 - 6\vec{v}_4$

6. $\langle x, y, z\rangle = \langle 4-3z, 7-5z, z\rangle = \langle 4, 7, 0\rangle + z\langle -3, -5, 1\rangle$
This is the equation of a line passing through $(4, 7, 0)$, with direction vector $\langle -3, -5, 1\rangle$.

7. \$1.50 per shirt, \$5 per pair of slacks, and \$7 per jacket.

8. 1 kilogram of Barley, 3 kilograms of Oats, and 2 kilogram of Soy.

9. The rref is $\begin{bmatrix} 1 & 0 & -\frac{4}{5} & \frac{159}{5} \\ 0 & 1 & \frac{9}{5} & \frac{331}{5} \end{bmatrix}$, so $d = (159+4p)/5$ and $n = (331-9p)/5$.
The solution with the smallest number of pennies has $p = 4$, $n = 59$, and $d = 35$. Since we want $n \geq 0$, we need $p \leq 331/9 \approx 36.8$. The solution with the largest number of pennies has $p = 34$, $n = 5$ and $d = 59$.

1.6 Exercises

1. ***Assisted Computation***:

a. (i) underdetermined; (ii) consistent; (iii) $\vec{x} = \langle 6-5x_4, -8+3x_4, 10-7x_4, x_4\rangle$; (iv) infinite number of solutions

b. (i) overdetermined; (ii) consistent; (iii) $\vec{x} = \langle 3, 1, -2\rangle$; (iv) unique solution

c. (i) square; (ii) inconsistent.

d. (i) square; (ii) consistent; (iii) $\vec{x} = \langle 10-7x_4, 6-5x_4, -8+3x_4, x_4\rangle$, where x_4 is free; (iv) infinite number of solutions

e. (i) overdetermined; (ii) inconsistent.

f. (i) underdetermined; (ii) consistent;
(iii) $\vec{x} = \langle 5-3x_5, -8+2x_5, 6-4x_5, 8-3x_5, x_5\rangle$, where x_5 is free;
(iv) infinite number of solutions

g. (i) underdetermined; (ii) consistent; (iii) $\vec{x} = \langle -6-7x_4, 2+3x_4, 3-5x_4, x_4, -9\rangle$, where x_4 is free; (iv) infinite number of solutions

h. (i) underdetermined; (ii) consistent;
(iii) $\vec{x} = \langle 3-7x_3-2x_6, -5-5x_3-6x_6, x_3, 4+3x_6, -6-4x_6, x_6\rangle$, where x_3 and x_6 are free; (iv) infinite number of solutions

i. (i) underdetermined; (ii) consistent;
(iii) $\vec{x} = \langle 3-5x_4-6x_5, 2+3x_4+8x_5, -6-7x_4-10x_5, x_4, x_5, -9\rangle$, where x_4 and x_5 are free; (iv) infinite number of solutions

j. (i) overdetermined; (ii) consistent; (iii) $\vec{x} = \langle 5-7x_3, 2-5x_3, x_3, 5\rangle$, where x_3 is free; (iv) infinite number of solutions

2. Answers:
   a. (iii) square; (iv) consistent; (v) $\vec{x} = \langle 5, 3 \rangle$; (vi) unique solution.
   b. (iii) overdetermined; (iv) consistent; (v) $\vec{x} = \langle 3, -5 \rangle$; (vi) unique solution.
   c. (iii) overdetermined; (iv) inconsistent.
   d. (iii) overdetermined; (iv) inconsistent.
   e. (iii) underdetermined; (iv) consistent; (v) $\vec{x} = \langle 5 - 3z, -8 + 7z, z \rangle$; (vi) infinite number of solutions.
   f. (iii) underdetermined; (iv) consistent; (v) $\vec{x} = \langle 4 - 2y + 5z, y, z \rangle$; (vi) infinite number of solutions.
   g. (iii) square; (iv) consistent; (v) $\vec{x} = \langle 5 + 2z, -4 - 5z, z \rangle$; (vi) infinite number of solutions.
   h. (iii) square; (iv) consistent; (v) $\vec{x} = \langle 6, 9, -5 \rangle$; (vi) unique solution.
3. Answers:
   a. $\vec{x} = \langle -5x_4, 3x_4, -7x_4, x_4 \rangle$, where x_4 is free.
   b. $\vec{x} = \langle -7x_3, -5x_3, x_3 \rangle$, where x_3 is free.

      Note: even though Exercise 1 (c) is inconsistent, the corresponding homogeneous system is consistent (reminder: any ***homogeneous*** system is ***always*** consistent).
   c. $\vec{x} = \langle -7x_4, 3x_4, -5x_4, 0 \rangle$, where x_4 is free.
   d. $\vec{x} = \langle -7x_3 - 2x_6, -5x_3 - 6x_6, x_3, 3x_6, -4x_6, x_6 \rangle$, where x_3 and x_6 are free.
   e. $\vec{x} = \langle -5x_4 - 6x_5, 3x_4 + 8x_5, -7x_4 - 10x_5, x_4, x_5, 0 \rangle$, where x_4 and x_5 are free.
   f. $\vec{x} = \langle -7x_3, -5x_3, x_3, 0 \rangle$, where x_3 is free.
4. a. $\begin{bmatrix} -14 \\ 77 \\ -46 \end{bmatrix}$ b. $\begin{bmatrix} 107 \\ 45 \\ -26 \end{bmatrix}$ c. $\begin{bmatrix} 11 \\ -16 \\ 40 \\ -43 \end{bmatrix}$ d. $\begin{bmatrix} -16 \\ 69 \\ -10 \\ 49 \end{bmatrix}$
5. Answers:
   a. $\vec{x} = \langle 7 - 5z, -4 + 2z, z \rangle$, where z is free.
   b. $\vec{x} = \langle -7 - 4x_3 - 9x_4, 6 + 2x_3 + 5x_4, x_3, x_4 \rangle$, where x_3 and x_4 are free.
   c. $\vec{x} = \langle 4 - 3x_4, -3 + 5x_4, -5 + 4x_4, x_4 \rangle$, where x_4 is free.
   d. $\vec{x} = \langle 8 - 7x_3 + 8x_5, -4 - 5x_3 - 6x_5, x_3, 7 + 9x_5, x_5 \rangle$, where x_3 and x_5 are free.
   e. $\vec{x} = \langle 3 - 7x_3 - 2x_6, -5 - 5x_3 - 6x_6, x_3, 4 + 3x_6, -6 - 4x_6, x_6 \rangle$, where x_3 and x_6 are free.
   f. $\vec{x} = \langle 3 - 7x_3 - 2x_6, -5 - 5x_3 - 6x_6, x_3, 4 + 3x_6, -6 - 4x_6, x_6 \rangle$, where x_3 and x_6 are free.

      Yes, it's exactly the same answer as part (e), and obviously it's not an accident. More on this later!

6. The system will have no solution if $r = -4$ and $s \neq \frac{7}{2}$. The system will have exactly one solution if $r \neq -4$ and s is ***any*** real number. The system will have an infinite number of solutions if $r = -4$ and $s = \frac{7}{2}$.

7. In all cases, x is a leading variable. The system will have no solution if $s = -8$ and $t \neq 4$. The system will have exactly one solution if $s \neq -8$, t is ***any*** real number, and $r \neq -6$. The system will have an infinite number of solutions involving exactly one free variable in two ways. First, if $s = -8$, $t = 4$, and $r \neq -6$, then y is a leading variable and z is a free variable. If $r = -6$, then z is automatically a leading variable because of the 2nd equation, and $z = -\frac{13}{10}$. This will satisfy the 3rd equation if and only if $(8+s)\left(-\frac{13}{10}\right) = t - 4$, so $10t + 13s = -144$. Thus, the second way is to have $r = -6$ and s and t any two real numbers satisfying $10t + 13s = -144$. In this case, y is a free variable. The system will never have an infinite number of solutions involving exactly two free variables.

11. a. False. b. False. c. True. d. False e. True. f. False. g. True. h. False.

Chapter Two Exercises

2.1 Exercises

1. ***Assisted Computation***:
   a. (i) dependent; (ii) $S' = \{\vec{v}_1, \vec{v}_2\}$; (iii) $\vec{v}_3 = 5\vec{v}_1 - 2\vec{v}_2$
   b. (i) dependent; (ii) $S' = \{\vec{v}_1, \vec{v}_3\}$; (iii) $\vec{v}_2 = -3\vec{v}_1$; $\vec{v}_4 = 2\vec{v}_1 - \vec{v}_3$
   c. (i) independent
   d. (i) dependent; (ii) $S' = \{\vec{v}_1, \vec{v}_2\}$; (iii) $\vec{v}_3 = 3\vec{v}_1 + 5\vec{v}_2$
   e. (i) dependent; (ii) $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$; (iii) $\vec{v}_4 = 7\vec{v}_1 + 5\vec{v}_2 - 3\vec{v}_3$
   f. (i) dependent; (ii) $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$; (iii) $\vec{v}_4 = 3\vec{v}_1 - 5\vec{v}_2 - 4\vec{v}_3$, and $\vec{v}_5 = 4\vec{v}_1 - 3\vec{v}_2 - 5\vec{v}_3$
   g. (i) dependent; (ii) $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\}$; (iii) $\vec{v}_3 = 7\vec{v}_1 + 5\vec{v}_2$, and $\vec{v}_6 = 2\vec{v}_1 + 6\vec{v}_2 - 3\vec{v}_4 + 4\vec{v}_5$
   h. (i) dependent; (ii) $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_6\}$; (iii) $\vec{v}_4 = 5\vec{v}_1 - 3\vec{v}_2 + 7\vec{v}_3$, and $\vec{v}_5 = 6\vec{v}_1 - 8\vec{v}_2 + 10\vec{v}_3$
   i. (i) dependent; (ii) $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_6\}$; (iii) $\vec{v}_3 = 4\vec{v}_1 + 3\vec{v}_2$, and $\vec{v}_5 = 6\vec{v}_1 + 4\vec{v}_2 + 3\vec{v}_4$
   j. (i) dependent; (ii) $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\}$; (iii) $\vec{v}_3 = 7\vec{v}_1 + 5\vec{v}_2$, and $\vec{v}_6 = 2\vec{v}_1 + 6\vec{v}_2 - 3\vec{v}_4 + 4\vec{v}_5$
2. Answers:
   a. dependent; $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = -2\vec{v}_1 + \vec{v}_2$
   b. independent
   c. independent
   d. dependent; $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = -\frac{2}{5}\vec{v}_1 + \frac{1}{5}\vec{v}_2$
   e. dependent; $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$; $\vec{v}_3 = 4\vec{v}_1 + 7\vec{v}_2$
   f. dependent; $S' = \{\vec{v}_1, \vec{v}_2\}$; $\vec{v}_3 = \frac{3}{5}\vec{v}_1 + \frac{1}{5}\vec{v}_2$
   g. dependent; $S' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$; $\vec{v}_4 = 2\vec{v}_1 + 3\vec{v}_2 - 4\vec{v}_3$
3. Answers:
   a. (i) $\vec{x} = \langle 3x_2 - 2x_4, x_2, x_4, x_4 \rangle$; (ii) $2\vec{v}_2 + 3\vec{v}_3 + 3\vec{v}_4 = \vec{0}_3$
   b. (i) $\vec{x} = \langle -3x_4 - 4x_5, 5x_4 + 3x_5, 4x_4 + 5x_5, x_4, x_5 \rangle$; (ii) $-\vec{v}_1 - 13\vec{v}_2 - 5\vec{v}_4 + 4\vec{v}_5 = \vec{0}_4$
   c. (i) $\vec{x} = \langle -7x_3 - 2x_6, -5x_3 - 6x_6, x_3, 3x_6, -4x_6, x_6 \rangle$; (ii) the given subset is linearly independent.
   d. (i) $\vec{x} = \langle -5x_4 - 6x_5, 3x_4 + 8x_5, -7x_4 - 10x_5, x_4, x_5, 0 \rangle$;
   (ii) $22\vec{v}_1 + 26\vec{v}_3 - 8\vec{v}_4 + 3\vec{v}_5 = \vec{0}_4$
   e. (i) $\vec{x} = \langle -5x_4 - 6x_5, 3x_4 + 8x_5, -7x_4 - 10x_5, x_4, x_5, 0 \rangle$;

(ii) $22\vec{v}_2 - 8\vec{v}_3 - 6\vec{v}_4 + 5\vec{v}_5 = \vec{0}_4$

f. (i) $\vec{x} = \langle -4x_3 - 6x_5, -3x_3 - 4x_5, x_3, -3x_5, x_5, 0 \rangle$; (ii) $\vec{v}_2 - 3\vec{v}_3 - 6\vec{v}_4 + 2\vec{v}_5 = \vec{0}_4$

g. (i) $\vec{x} = \langle -4x_3 - 6x_5, -3x_3 - 4x_5, x_3, -3x_5, x_5, 0 \rangle$; (ii) the given subset is linearly independent.

h. (i) $\vec{x} = \langle -7x_3 - 2x_6, -5x_3 - 6x_6, x_3, 3x_6, -4x_6, x_6 \rangle$; (ii) the given subset is linearly independent.

4. $c = 22$.

2.2 Exercises

1. ***Subspaces of*** $\mathbb{R}^2$ ***and*** $\mathbb{R}^3$:
   - a. $\{\langle 7, 5 \rangle\}$
   - b. $\{\langle 3, -\sqrt{2} \rangle\}$ or $\{\langle -3, \sqrt{2} \rangle\}$
   - c. $\{\langle 3\sqrt{2}, -2 \rangle\}$ or $\{\langle -3\sqrt{2}, 2 \rangle\}$
   - d. $\{\langle 7, 0, 3 \rangle, \langle 0, 4, 7 \rangle\}$ is one possible answer. There are two more answers, which you can get by replacing one of these vectors with $\langle 4, 0, -3 \rangle$.
   - e. $\{\langle 5, 0, 2 \rangle, \langle 0, 1, 0 \rangle\}$
   - f. It does not pass through the origin.
   - g. It does not pass through the origin.
   - h. No. It is not a line through the origin, nor is it one of the trivial subspaces of $\mathbb{R}^2$.
2. ***Assisted Computation***:
   - a. $\{\vec{w}_1, \vec{w}_2\}$; $dim(W) = 2$
   - b. $\{\vec{w}_1, \vec{w}_3\}$; $dim(W) = 2$
   - c. $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$; $dim(W) = 3$
   - d. $\{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $dim(W) = 3$
   - e. $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$; $dim(W) = 3$
   - f. $\{\vec{w}_1, \vec{w}_2, \vec{w}_5\}$; $dim(W) = 3$
   - g. $\{\vec{w}_1, \vec{w}_2, \vec{w}_4, \vec{w}_5\}$; $dim(W) = 4$
   - h. $\{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $dim(W) = 3$
3. Use the ***Minimizing Theorem*** (Basis for a Subspace Version) to find a basis for the subspace $W = Span(S)$, for each of the sets S below. State $dim(W)$. Use technology if permitted by your instructor.
   - a. $\{\vec{w}_1, \vec{w}_2\}$; $dim(W) = 2$
   - b. $\{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $dim(W) = 3$
   - c. $\{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $dim(W) = 3$
   - d. $\{\vec{w}_1, \vec{w}_2\}$; $dim(W) = 2$
   - e. $\{\vec{w}_1, \vec{w}_2, \vec{w}_5\}$; $dim(W) = 3$

f. $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$; $dim(W) = 3$

g. $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$; $dim(W) = 3$

h. $\{\vec{w}_1, \vec{w}_2, \vec{w}_5\}$; $dim(W) = 3$

i. $\{\vec{w}_1, \vec{w}_2, \vec{w}_4, \vec{w}_5\}$; $dim(W) = 4$

j. $\{\vec{w}_1, \vec{w}_2, \vec{w}_5\}$; $dim(W) = 3$

k. $\{\vec{w}_1, \vec{w}_2, \vec{w}_4, \vec{w}_5\}$; $dim(W) = 4$

l. $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4, \vec{w}_5\}$; $dim(W) = 5$

2.3 Exercises

1. ***Assisted Computation***:

a. (i) $\{\langle 1, 0, 4, 5\rangle, \langle 0, 1, -2, -3\rangle\}$; $\{\langle 3, 5, 16\rangle, \langle 2, 7, 29\rangle\}$; $\{\langle -4, 2, 1, 0\rangle, \langle -5, 3, 0, 1\rangle\}$; (ii) $\{\langle 3, -5, 1\rangle\}$; (iii) $rank(A) = 2$; $nullity(A) = 2$; $nullity(A^\top) = 1$; (iv) not full rank

b. (i) $\{\langle 1, 0, 4, 0\rangle, \langle 0, 1, -3, 0\rangle, \langle 0, 0, 0, 1\rangle\}$; $\{\langle 5, -4, 3\rangle, \langle 6, -7, 2\rangle, \langle 1, 2, 3\rangle\}$; $\{\langle -4, 3, 1, 0\rangle\}$; (ii) no basis; (iii) $rank(A) = 3$; $nullity(A) = 1$; $nullity(A^\top) = 0$; (iv) full rank

c. (i) $\{\langle 1, 0, 4, 0, 6\rangle, \langle 0, 1, -3, 0, -3\rangle, \langle 0, 0, 0, 1, -4\rangle\}$; $\{\langle 5, 4, 3\rangle, \langle 6, 7, 2\rangle, \langle 1, -2, 3\rangle\}$; $\{\langle -4, 3, 1, 0, 0\rangle, \langle -6, 3, 0, 4, 1\rangle\}$; (ii) no basis; (iii) $rank(A) = 3$; $nullity(A) = 2$; $nullity(A^\top) = 0$; (iv) full rank

d. (i) $\{\langle 1, 0, 4, 0, 2\rangle, \langle 0, 1, -3, 0, -5\rangle, \langle 0, 0, 0, 1, 7\rangle\}$; $\{\langle 3, 5, 1, 4\rangle, \langle 4, 7, 2, 3\rangle, \langle 3, 4, -1, 2\rangle\}$; $\{\langle -4, 3, 1, 0, 0\rangle, \langle -2, 5, 0, -7, 1\rangle\}$; (ii) $\{\langle 3, -2, 1, 0\rangle\}$; (iii) $rank(A) = 3$; $nullity(A) = 2$; $nullity(A^\top) = 1$; (iv) not full rank

e. (i) $\{\langle 1, 0, 5, 0, -8\rangle, \langle 0, 1, -7, 0, 3\rangle, \langle 0, 0, 0, 1, 7\rangle\}$; $\{\langle 4, 6, 17, 28\rangle, \langle 2, 3, 8, 13\rangle, \langle 6, 9, 29, 49\rangle\}$; $\{\langle -5, 7, 1, 0, 0\rangle, \langle 8, -3, 0, -7, 1\rangle\}$; (ii) $\{\langle 9, -5, -2, 1\rangle\}$; (iii) $rank(A) = 3$; $nullity(A) = 2$; $nullity(A^\top) = 1$; (iv) not full rank

f. (i) $\{\langle 1, 0, -7, 0, -9\rangle, \langle 0, 1, 4, 0, 3\rangle, \langle 0, 0, 0, 1, 2\rangle, \}$; $\{\langle 4, 2, 5, 7, 10\rangle, \langle 11, 5, 12, 9, 19\rangle, \langle 9, 4, 10, 8, 17\rangle, \}$; $\{\langle 7, -4, 1, 0, 0\rangle, \langle 9, -3, 0, -2, 1\rangle\}$; (ii) $\{\langle 6, -3, -5, 1, 0\rangle, \langle 3, 4, -6, 0, 1\rangle\}$; (iii) $rank(A) = 3$; $nullity(A) = 2$; $nullity(A^\top) = 2$; (iv) not full rank

g. (i) $\{\langle 1, 0, 4, 0, 0\rangle, \langle 0, 1, -5, 0, 0\rangle, \langle 0, 0, 0, 1, 0\rangle, \langle 0, 0, 0, 0, 1\rangle\}$; $\{\langle 3, 1, 0, -1, -4\rangle, \langle 2, 1, 2, -6, -7\rangle, \langle -1, -1, -3, 7, 8\rangle, \langle -1, 0, 4, -13, -6\rangle\}$; $\{\langle 4, -5, 1, 0, 0\rangle\}$; (ii) $\{\langle 3, -8, 4, 1, 0\rangle\}$; (iii) $rank(A) = 4$; $nullity(A) = 1$; $nullity(A^\top) = 1$; (iv) not full rank

h. (i) $\{\langle 1, 0, 9, 0, 5, 3\rangle, \langle 0, 1, -4, 0, 2, 5\rangle, \langle 0, 0, 0, 1, -4, -6\rangle\}$; $\{\langle 4, 2, 4, -2, 1\rangle, \langle 9, 4, 6, -7, 2\rangle, \langle 11, 5, 8, -8, 2\rangle\}$; $\{\langle -9, 4, 1, 0, 0, 0\rangle, \langle -5, -2, 0, 4, 1, 0\rangle, \langle -3, -5, 0, 6, 0, 1\rangle\}$;

(ii) $\{\langle 2,-6, 1, 0, 0\rangle, \langle 3,-5, 0, 1, 0\rangle\}$;
(iii) $rank(A) = 3$; $nullity(A) = 3$; $nullity(A^\top) = 2$; (iv) not full rank

i. (i) $\{\langle 1, 0, 1, 0, 0\rangle, \langle 0, 1,-7, 0, 0\rangle, \langle 0, 0, 0, 1, 0\rangle, \langle 0, 0, 0, 0, 1\rangle\}$;
$\{\langle 3, 0, 12,-1, 12,-1\rangle, \langle 1,-1, 1,-1, 0, 0\rangle, \langle -2,-2,-14,-1,-17, 4\rangle, \langle 0, 5, 15, 4, 22,-6\rangle\}$
$\{\langle -1, 7, 1, 0, 0\rangle\}$; (ii) $\{\langle -4,-3, 1, 0, 0, 0\rangle, \langle -5,-2, 0,-3, 1, 0\rangle\}$;
(iii) $rank(A) = 4$; $nullity(A) = 1$; $nullity(A^\top) = 2$; (iv) not full rank

j. (i) $\{\langle 1, 0, 2, 0, 0, 5\rangle, \langle 0, 1, 3, 0, 0, 2\rangle, \langle 0, 0, 0, 1, 0, 7\rangle, \langle 0, 0, 0, 0, 1, 4\rangle\}$;
$\{\langle 3, 4, 1,-6,-1, 9\rangle, \langle 1,-3,-2, 1, 1, 2\rangle, \langle 0, 2, 1,-2, 2,-18\rangle, \langle -4,-5,-1, 9,-3, 18\rangle\}$;
$\{\langle -2,-3, 1, 0, 0, 0\rangle, \langle -5,-2, 0,-7,-4, 1\rangle\}$;
(ii) $\{\langle -2, 5,-8, 1, 0, 0\rangle, \langle -4, 3,-2, 0, 7, 1\rangle\}$;
(iii) $rank(A) = 4$; $nullity(A) = 2$; $nullity(A^\top) = 2$; (iv) not full rank

2. Answers:

a. (i) $\{\langle 1,-3, 0, 5\rangle, \langle 0, 0, 1,-3\rangle\}$; $\{\langle 2,-3,-2\rangle, \langle 5,-6, 1\rangle\}$;
$\{\langle 3, 1, 0, 0\rangle, \langle -5, 0, 3, 1\rangle\}$; (ii) $\{\langle -5,-4, 1\rangle\}$;
(iii) $rank(A) = 2$; $nullity(A) = 2$; $nullity(A^\top) = 1$; (iv) not full rank

b. (i) $\{\langle 1, 0, 7, 0, 0, 2\rangle, \langle 0, 1, 5, 0, 0, 6\rangle, \langle 0, 0, 0, 1, 0,-3\rangle, \langle 0, 0, 0, 0, 1, 4\rangle\}$;
$\{\langle 2,-5, 1, 3\rangle, \langle -3, 6, 0,-4\rangle, \langle -4, 2, 3, 1\rangle, \langle 2,-7, 4, 5\rangle\}$;
$\{\langle -7,-5, 1, 0, 0, 0\rangle, \langle -2,-6, 0, 3,-4, 1\rangle\}$; (ii) no basis;
(iii) $rank(A) = 4$; $nullity(A) = 2$; $nullity(A^\top) = 0$; (iv) full rank

c. (i) $\{\langle 1, 0, 0, 7\rangle, \langle 0, 1, 0, 5\rangle, \langle 0, 0, 1,-3\rangle\}$;
$\{\langle 3,-2,-4,7\rangle, \langle -4,3,2,-5\rangle, \langle 1,-2,-3,6\rangle, \}$; $\{\langle -7,-5, 3, 1\rangle\}$;
(ii) $\{\langle 1, 5, 7, 5\rangle\}$; (iii) $rank(A) = 3$; $nullity(A) = 1$; $nullity(A^\top) = 1$;
(iv) not full rank

d. (i) $\{\langle 1, 0, 4, 0, 6\rangle, \langle 0, 1, 3, 0, 4\rangle, \langle 0, 0, 0, 1, 3\rangle\}$;
$\{\langle 2,-5,-3, 3\rangle, \langle -3, 6, 1,-2\rangle, \langle -1, 5, 3,-2\rangle\}$;
$\{\langle -4,-3, 1, 0, 0\rangle, \langle -6,-4, 0,-3, 1\rangle\}$; (ii) $\{\langle -13,-4, 11, 13\rangle\}$;
(iii) $rank(A) = 3$; $nullity(A) = 2$; $nullity(A^\top) = 1$; (iv) not full rank

e. (i) $\{\langle 1, 0, 7, 0, 8, 5\rangle, \langle 0, 1,-4, 0, 5, 3\rangle, \langle 0, 0, 0, 1,-10,-7\rangle\}$;
$\{\langle 5,-3,-1, 6\rangle, \langle 8,-5,-3, 9\rangle, \langle 8,-5,-3, 10\rangle\}$;
$\{\langle -7, 4, 1, 0, 0, 0\rangle, \langle -8,-5, 0, 10, 1, 0\rangle, \langle -5,-3, 0, 7, 0, 1\rangle\}$;
(ii) $\{\langle -4,-7, 1, 0\rangle\}$; (iii) $rank(A) = 3$; $nullity(A) = 3$; $nullity(A^\top) = 1$;
(iv) not full rank

f. (i) $\{\langle 1, 0, 7, 0, 4\rangle, \langle 0, 1, 5, 0, 6\rangle, \langle 0, 0, 0, 1,-3\rangle\}$;
$\{\langle 2,-5,-4, 1, 3\rangle, \langle -3, 6, 3, 0, 3\rangle, \langle -4, 3, 1,-2, 6\rangle\}$;
$\{\langle -7,-5, 1, 0, 0\rangle, \langle -4,-6, 0, 3, 1\rangle\}$;
(ii) $\{\langle -1,-1, 1, 1, 0\rangle, \langle 2, 1,-1, 0, 1\rangle\}$;
(iii) $rank(A) = 3$; $nullity(A) = 2$; $nullity(A^\top) = 2$; (iv) not full rank

3. A is a 6×13 matrix.

10. a. False. b. True. c. False. d. False.

2.4 Exercises

1. ***Basic Computations:***

   a. $\|\vec{u}\| = \sqrt{119}$

   b. $\cos(\theta) = -7/(5\sqrt{29}) \approx -0.25997$
   $\cos^{-1}(-0.25997) \approx 1.8338$ radians$\approx$ 105.07 degrees

   c. $\|2\vec{u}\| + \|5\vec{v}\| = 2\sqrt{34} + 5\sqrt{65} \approx 51.973$ is bigger than
   $\|2\vec{u} + 5\vec{v}\| = \sqrt{941} \approx 30.676$. This verifies the Triangle Inequality

   d. $\cos(\theta) = \dfrac{37}{\sqrt{83}\sqrt{77}} \approx 0.46283$, and $\theta \approx \cos^{-1}(0.46283) \approx 1.0896$ radians

   e. $\sqrt{86}$

   f. $\cos(\theta) = \dfrac{5}{7\sqrt{23}} \approx 0.14894$, and $\theta \approx \cos^{-1}(0.14894) \approx 1.4213$ radians

   g. $\cos(\theta) = \dfrac{1}{\sqrt{3}} = 0.57735$, and $\theta \approx \cos^{-1}(0.57735) \approx 0.95532$ radians
   $\theta \approx \cos^{-1}(0.57735) \approx 0.95532$ radians or 54.736 degrees

   h. We can use the vectors $\langle 2, 3, 5\rangle$, $\langle 2, 0, 0\rangle$, $\langle 0, 3, 0\rangle$, and $\langle 0, 0, 5\rangle$.
   $\cos(\alpha) = \dfrac{2}{\sqrt{38}} \approx 0.32444$
   $\alpha \approx \cos^{-1}(0.32444) \approx 1.2404$ radians or 71.07 degrees
   $\cos(\beta) = \dfrac{3}{\sqrt{38}} \approx 0.48666$
   $\beta \approx \cos^{-1}(0.48666) \approx 1.0625$ radians or 60.88 degrees
   $\cos(\gamma) = \dfrac{5}{\sqrt{38}} \approx 0.81111$
   $\gamma \approx \cos^{-1}(0.81111) \approx 0.62475$ radians or 35.8 degrees

2. ***Applying the Properties:***

   a. $(3\vec{u} - 8\vec{v}) \circ (3\vec{u} + 8\vec{v}) = -2911$

   b. $\|4\vec{u} + 11\vec{v}\| = \sqrt{4569}$

   c. $\|7\vec{u} - 3\vec{v}\| = \sqrt{7837}$

   d. $\vec{u} \circ \vec{v} = 24$

   e. $\|\vec{u}\| = 29$, $\|\vec{v}\| = 13$, and $\|3\vec{u} - 8\vec{v}\| = \sqrt{34945}$

3. ***Parallel Planes:***

   a. $6x - 5y + 2z = -15$

   b. $2x + 5y - 9z = 40$

4. ***The Cross Product:***

   a. $\vec{u} \times \vec{v} = \langle 11, 37, 54\rangle$, and $-\vec{u} \times \vec{v} = \langle -11, -37, -54\rangle$.

   b. both dot products are 0, so $\vec{u}$ and $\vec{v}$ are orthogonal to $\vec{u} \times \vec{v}$.

5. ***Intersecting Lines:***

   a. $(13, 3, 6)$

   b. $5x + 13y + z = 110$

6. ***Orthogonal Lines***:

   a. they intersect at $(2, 5, -3)$; the dot product of the two direction vectors is 0; the equation of the plane is $x + y - z = 10$

   b. $\langle x, y, z\rangle = \langle 8 + 21t, 39 + 24t, -11 - 15t\rangle$, or reduce direction vector to:
   $\langle x, y, z\rangle = \langle 8 + 7t, 39 + 8t, -11 - 5t\rangle$

7. c. $29x + 10y - 16z = 125$

8. ***Skew Lines***:

   a. they have no point of intersection, and the direction vectors are not parallel.

   b. $\langle 7, 11, -13\rangle$

   c. $7x + 11y - 13z = 46$, and $7x + 11y - 13z = 104$

9. ***Orthogonal Planes***:

   a. $\langle x, y, z\rangle = \left\langle \frac{9}{22} - \frac{43}{22}t, -\frac{21}{22} - \frac{17}{22}t, t\right\rangle$ or $\langle x, y, z\rangle = \left\langle \frac{9}{22} - 43t, -\frac{21}{22} - 17t, 22t\right\rangle$

   b. $-x + y + 4z = -25$

   c. $15x + 13y + 10z = 68$

   d. $\langle x, y, z\rangle = \langle 3 - 2t, 1, 1 + 3t\rangle$
   $3(3 - 2t) - 5 + 2(1 + 3t) = 6$, and
   $15(3 - 2t) + 13 + 10(1 + 3t) = 68$, so both planes check.

10. ***Orthogonal Line and Plane Pairs***:

    a. $\vec{d} = \langle 2, -6, 8\rangle$ is parallel to $\vec{n} = \langle 1, -3, 4\rangle$

    b. $8x + 5y - 4z = 2$; point of intersection: $\left(\frac{118}{105}, \frac{62}{21}, \frac{571}{105}\right)$

    c. $\langle x, y, z\rangle = \langle 5, -2, 1\rangle + t\langle 3, 7, -4\rangle$; point of intersection: $\left(\frac{397}{74}, -\frac{85}{74}, \frac{19}{37}\right)$

11. ***Parallel Lines and Planes***:

    a. $2(-2 + 8t) - 4(1 + 5t) - (7 - 4t) = -15$, not 3, so L and Π_1 do not intersect.

    b. $x + 2z = 12$

12. False. The correct second phrase is "$\vec{u}$ and $\vec{v}$ are orthogonal to each other."

2.5 Exercises

1. ***Assisted Computation***:

   a. (i) $\{\langle 3, -1, 3, 1\rangle, \langle -7, 3, -1, 1\rangle\}$; (ii) $\{\langle -4, -9, 1, 0\rangle, \langle -2, -5, 0, 1\rangle\}$;
   (iii) $\{\langle 1, 0, 4, 2\rangle, \langle 0, 1, 9, 5\rangle\}$; (iv) $dim(W) = 2$, and $dim(W^{\perp}) = 2$; $2 + 2 = 4$.

   b. (i) $\{\langle 3, -2, 4, 2\rangle, \langle -5, 5, -7, 9\rangle, \langle 2, -3, 5, 5\rangle\}$; (ii) $\{\langle 4, -9, -8, 1\rangle\}$;
   (iii) $\{\langle 1, 0, 0, -4\rangle, \langle 0, 1, 0, 9\rangle, \langle 0, 0, 1, 8\rangle\}$;
   (iv) $dim(W) = 3$, and $dim(W^{\perp}) = 1$; $3 + 1 = 4$.

   c. (i) $\{\langle 3, -5, 2, -3\rangle, \langle -2, 5, -3, 12\rangle, \langle 4, -7, 5, -14\rangle\}$; (ii) $\{\langle -5, -2, 4, 1\rangle\}$;
   (iii) $\{\langle 1, 0, 0, 5\rangle, \langle 0, 1, 0, 2\rangle, \langle 0, 0, 1, -4\rangle\}$;
   (iv) $dim(W) = 3$, and $dim(W^{\perp}) = 1$; $3 + 1 = 4$.

d. (i) $\{\langle 3,-2,-2,4\rangle, \langle -5,4,8,-7\rangle, \langle 2,-3,-13,5\rangle\}$; (ii) $\{\langle -4,-7,1,0\rangle\}$;
(iii) $\{\langle 1,0,4,0\rangle, \langle 0,1,7,0\rangle, \langle 0,0,0,1\rangle\}$;
(iv) $dim(W) = 3$, and $dim(W^{\perp}) = 1$; $3+1 = 4$.

e. (i) $\{\langle 5,6,0,1,9\rangle, \langle 4,7,-11,-2,14\rangle, \langle 3,2,8,3,-1\rangle\}$;
(ii) $\{\langle -6,5,1,0,0\rangle, \langle -5,2,0,4,1\rangle\}$;
(iii) $\{\langle 1,0,6,0,5\rangle, \langle 0,1,-5,0,-2\rangle, \langle 0,0,0,1,-4\rangle\}$;
(iv) $dim(W) = 3$, and $dim(W^{\perp}) = 2$; $3+2 = 5$.

f. (i) $\{\langle 3,4,-4,2,1\rangle, \langle 5,7,-6,11,8\rangle, \langle -4,-3,3,-5,-6\rangle\}$;
(ii) $\{\langle -2,-7,-8,1,0\rangle, \langle -3,-5,-7,0,1\rangle\}$;
(iii) $\{\langle 1,0,0,2,3\rangle, \langle 0,1,0,7,5\rangle, \langle 0,0,1,8,7\rangle\}$;
(iv) $dim(W) = 3$, and $dim(W^{\perp}) = 2$; $3+2 = 5$.

g. (i) $\{\langle 2,-3,1,6,3,-5\rangle, \langle -5,7,-4,-8,3,-14\rangle, \langle 3,-2,9,1,6,17\rangle\}$;
(ii) $\{\langle -5,-3,1,0,0,0\rangle, \langle -6,-7,0,-2,1,0\rangle, \langle -5,3,0,4,0,1\rangle\}$;
(iii) $\{\langle 1,0,5,0,6,5\rangle, \langle 0,1,3,0,7,-3\rangle, \langle 0,0,0,1,2,-4\rangle\}$;
(iv) $dim(W) = 3$, and $dim(W^{\perp}) = 3$; $3+3 = 6$.

2. Answers:

a. (i) $\begin{bmatrix} -1 & 2 \\ 1 & -3 \\ 1 & -8 \\ 1 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$; (iii) $\begin{bmatrix} 1 & 0 & 5 & -4 \\ 0 & 1 & 6 & -3 \end{bmatrix}$;
(iv) $\{\langle -1,1,1,1\rangle, \langle 2,-3,-8,1\rangle\}$; (v) $\{\langle -5,-6,1,0\rangle, \langle 4,3,0,1\rangle\}$;
(vi) $\{\langle 1,0,5,-4\rangle, \langle 0,1,6,-3\rangle\}$; (vii) $dim(W) = 2$, and $dim(W^{\perp}) = 2$; $2+2 = 4$.

b. (i) $\begin{bmatrix} 3 & -4 & 2 \\ -1 & 3 & 1 \\ 3 & 11 & 17 \\ 8 & 1 & 17 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; (iii) $\begin{bmatrix} 1 & 0 & 4 & 5 \\ 0 & 1 & 9 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;
(iv) $\{\langle 3,-1,3,8\rangle, \langle -4,3,11,1\rangle\}$; (v) $\{\langle -4,-9,1,0\rangle, \langle -5,-7,0,1\rangle\}$;
(vi) $\{\langle 1,0,4,5\rangle, \langle 0,1,9,7\rangle\}$; (vii) $dim(W) = 2$, and $dim(W^{\perp}) = 2$; $2+2 = 4$.

c. (i) $\begin{bmatrix} 3 & -5 & 2 \\ -2 & 4 & -3 \\ -2 & 8 & -13 \\ 4 & -7 & 5 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; (iii) $\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;
(iv) $\{\langle 3,-2,-2,4\rangle, \langle -5,4,8,-7\rangle, \langle 2,-3,-13,5\rangle\}$; (v) $\{\langle -4,-7,1,0\rangle\}$;
(vi) $\{\langle 1,0,4,0\rangle, \langle 0,1,7,0\rangle, \langle 0,0,0,1\rangle\}$; (vii) $dim(W) = 3$, and $dim(W^{\perp}) = 1$; $3+1 = 4$.

d. (i) $\begin{bmatrix} 3 & -2 & 4 & 14 \\ -2 & 3 & -5 & -9 \\ 2 & 3 & -6 & 5 \\ -5 & 1 & 7 & -3 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; (iii) $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

(iv) $\{\langle 3, -2, 2, -5 \rangle, \langle -2, 3, 3, 1 \rangle, \langle 4, -5, -6, 7 \rangle\}$; (v) $\{\langle -7, -9, 4, 1 \rangle\}$;

(vi) $\{\langle 1, 0, 0, 7 \rangle, \langle 0, 1, 0, 9 \rangle, \langle 0, 0, 1, -4 \rangle\}$; (vii) $dim(W) = 3$, and $dim(W^{\perp}) = 1$; $3 + 1 = 4$.

e. (i) $\begin{bmatrix} 3 & 5 & -9 \\ 4 & 7 & -13 \\ -4 & -6 & 10 \\ 4 & 1 & 5 \\ -7 & -4 & -2 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; (iii) $\begin{bmatrix} 1 & 0 & -4 & 24 & -33 \\ 0 & 1 & 2 & -17 & 23 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$;

(iv) $\{\langle 3, 4, -4, 4, -7 \rangle, \langle 5, 7, -6, 1, -4 \rangle\}$;

(v) $\{\langle 4, -2, 1, 0, 0 \rangle, \langle -24, 17, 0, 1, 0 \rangle, \langle 33, -23, 0, 0, 1 \rangle\}$;

(vi) $\{\langle 1, 0, -4, 24, -33 \rangle, \langle 0, 1, 2, -17, 23 \rangle\}$;

(vii) $dim(W) = 2$, and $dim(W^{\perp}) = 3$; $2 + 3 = 5$.

f. (i) $\begin{bmatrix} 5 & 4 & 1 \\ 6 & 7 & -2 \\ 7 & -1 & 11 \\ 4 & 6 & -3 \\ 2 & 7 & -5 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; (iii) $\begin{bmatrix} 1 & 0 & 5 & 0 & 4 \\ 0 & 1 & -3 & 0 & -9 \\ 0 & 0 & 0 & 1 & 9 \end{bmatrix}$;

(iv) $\{\langle 5, 6, 7, 4, 2 \rangle, \langle 4, 7, -1, 6, 7 \rangle, \langle 1, -2, 11, -3, -5 \rangle\}$;

(v) $\{\langle -5, 3, 1, 0, 0 \rangle, \langle -4, 9, 0, -9, 1 \rangle\}$;

(vi) $\{\langle 1, 0, 5, 0, 4 \rangle, \langle 0, 1, -3, 0, -9 \rangle, \langle 0, 0, 0, 1, 9 \rangle\}$;

(vii) $dim(W) = 3$, and $dim(W^{\perp}) = 2$; $3 + 2 = 5$.

3. a. Yes. b. Yes. c. No. d. Yes. e. No. f. No.

4. a. Yes. b. Yes. c. No. d. Yes. e. No.

13. a. True. b. True. c. False. d. True. e. False. f. True. g. True. h. True. i. False. j. True. k. False. l. True. n. False. o. True. p. True. q. False. r. False. s. True. t. True. u. True.

14. a. False. b. True. c. False. d. False. e. True. f. False. g. False. h. True. i. False. j. True. k. False. l. True. m. False. n. True. o. False. p. False. q. False. r. True. s. True. t. False.

2.6 Exercises

1. ***Interpreting the RREF*:**
   a. $\langle -5, 8, 0\rangle + x_3\langle -9, 4, 1\rangle$
   b. $\langle 2, 0, -7\rangle + x_2\langle 4, 1, 0\rangle$
   c. $\langle 5, 6, 0, -4\rangle + x_3\langle -3, 2, 1, 0\rangle$
   d. $\langle 3, 0, -2, 0\rangle + x_3\langle 5, -4, 7, 1\rangle$
   e. $\langle -7, 3, -9, 0, 0\rangle + x_4\langle -5, 4, -2, 1, 0\rangle + x_5\langle 3, 0, -6, 0, 1\rangle$
   f. $\langle 5, -2, 0, -9, 0\rangle + x_3\langle -5, 6, 1, 0, 0\rangle + x_5\langle 0, -3, 0, 2, 1\rangle$
   g. $\langle -2, 0, 6, 7\rangle + x_2\langle -3, 1, 0, 0\rangle$
   h. $\langle -5, 2, 4, -1, 0\rangle + x_5\langle -6, 3, -2, -8, 1\rangle$
   i. $\langle -2, 5, 0, 6, 0\rangle + x_3\langle -3, 7, 1, 0, 0\rangle + x_5\langle -4, -2, 0, 9, 1\rangle$
   j. $\langle 3, 0, 0, -2, 4, 0\rangle + x_3\langle 9, -6, 1, 0, 0, 0\rangle + x_6\langle -5, 3, 0, -8, -2, 1\rangle$
2. ***Assisted Computation*:**
   a. (i) $\langle 5,-3,0\rangle + x_3\langle -4,2,1\rangle$; (ii) $\vec{b} = 5\vec{c}_1 - 3\vec{c}_2$; (iii) dependent
      (iv) Equation (3) = −3 × Equation (1) + 5 × Equation (2)
      (v) not full-rank
   b. (i) $\langle 6,-3,0,-4\rangle + x_3\langle -4,3,1,0\rangle$; (ii) $\vec{b} = 6\vec{c}_1 - 3\vec{c}_2 - 4\vec{c}_4$; (iii) independent;
      (v) full-rank
   c. (i) $\langle -4,6,9,0\rangle + x_4\langle -5,4,7,1\rangle$; (ii) $\vec{b} = -4\vec{c}_1 + 6\vec{c}_2 + 9\vec{c}_3$; (iii) dependent;
      (iv) Equation (3) = 2 × Equation (1) + Equation (2)
      (v) not full-rank
   d. (i) $\langle 37,-26,0,0\rangle + x_3\langle -4,3,1,0\rangle + x_4\langle -5,3,0,1\rangle$; (ii) $\vec{b} = 37\vec{c}_1 - 26\vec{c}_2$;
      (iii) dependent
      (iv) Equation (3) = 5 × Equation (1) −4 × Equation (2)
      Equation (4) = −3 × Equation (1) + 2 × Equation (2)
      (v) not full-rank
   e. (i) $\langle -1,2,0,3,0\rangle + x_3\langle -7,-5,1,0,0\rangle + x_5\langle -2,-3,0,1,1\rangle$
      (ii) $\vec{b} = -\vec{c}_1 + 2\vec{c}_2 + 3\vec{c}_4$; (iii) dependent
      (iv) Equation (4) = 3 × Equation (1) + 4 × Equation (2) + 2 × Equation (3)
      (v) not full-rank
   f. (i) $\langle -9,3,0,2\rangle + x_3\langle 7,-4,1,0\rangle$; (ii) $\vec{b} = -9\vec{c}_1 + 3\vec{c}_2 + 2\vec{c}_4$
      (iii) dependent
      (iv) Equation (4) = −6 × Equation (1) +3 × Equation (2) + 5 × Equation (3)
      Equation (5) = −3 × Equation (1) – 4 × Equation (2) + 6 × Equation (3)
      (v) not full-rank
   g. (i) $\langle 3,5,0,-6,0\rangle + x_3\langle -9,4,1,0,0\rangle + x_5\langle -5,-2,0,4,1\rangle$
      (ii) $\vec{b} = 3\vec{c}_1 + 5\vec{c}_2 - 6\vec{c}_4$; (iii) dependent

(iv) Equation (3) = −2 × Equation (1) + 6 × Equation (2)
Equation (4) = −3 × Equation (1) + 5 × Equation (2)
(v) not full-rank

h. (i) $\langle 5,2,7,0,4\rangle + x_4\langle -2,-3,-1,1,0\rangle$
(ii) $\vec{b} = 5\vec{c}_1 + 2\vec{c}_2 + 7\vec{c}_3 + 4\vec{c}_5$; (iii) dependent
(iv) Equation (4) = 2 × Equation (1) − 5 × Equation (2) + 8 × Equation (3)
Equation (6) = 4 × Equation (1) − 3 × Equation (2) + 2 × Equation (3) − 7 × Equation (5)
(v) not full-rank

3. Answers:

a. (i) $\begin{bmatrix} 1 & 0 & 7 & -9 \\ 0 & 1 & 5 & -7 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$;
(ii) $\langle 9,7,0\rangle + x_3\langle -7,-5,1\rangle$; (iii) independent; (iv) full-rank

b. (i) $\begin{bmatrix} 1 & 0 & 5 & -3 & 3 \\ 0 & 1 & 9 & -4 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;
(ii) $\langle 3,7,0,0\rangle + x_3\langle -5,-9,1,0\rangle + x_4\langle 3,4,0,1\rangle$; (iii) dependent;
Equation (3) = 2 × Equation (1) − 3 × Equation (2)
(iv) not full-rank

c. (i) $\begin{bmatrix} 1 & 0 & 4 & 0 & 4 & 5 \\ 0 & 1 & -3 & 0 & 9 & 7 \\ 0 & 0 & 0 & 1 & -8 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;
(ii) $\langle 5,7,0,-7,0\rangle + x_3\langle -4,3,1,0,0\rangle + x_5\langle -4,-9,0,8,1\rangle$; (iii) dependent;
Equation (3) = 3 × Equation (1) + 2 × Equation (2)
(iv) not full-rank

d. (i) $\begin{bmatrix} 1 & 0 & 0 & -5 & 4 \\ 0 & 1 & 0 & 7 & -6 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$;

(ii) $\langle 4,-6,5,0\rangle + x_4\langle 5,-7,4,1\rangle$; (iii) dependent;

Equation (4) = Equation (1) + 2 × Equation (2) + 2 × Equation (3)

Equation (5) = −2 × Equation (1) − 2 × Equation (2) + Equation (3)

(iv) not full-rank

e. (i) $\begin{bmatrix} 1 & 0 & 0 & -7 & 0 & 8 \\ 0 & 1 & 0 & 6 & 0 & -4 \\ 0 & 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$;

(ii) $\langle 8,-4,3,0,9\rangle + x_4\langle 7,-6,-9,1,0\rangle$; (iii) dependent;

Equation (4) = 2 × Equation (1) + Equation (2) − Equation (3)

(iv) not full-rank

4. ***Subspaces of $\mathbb{R}^n$ Described in Set-Builder Notation:***

a. This is the *xz*-plane. $\{\langle 1, 0, 0\rangle, \langle 0, 0, 1\rangle\}$ is a basis for W, and $dim(W) = 2$.

b. W does not contain $\vec{\mathbf{0}}_3$.

c. This is the *x*-axis. $\{\langle 1, 0, 0\rangle\}$ is a basis for W, and $dim(W) = 1$.

d. W is not a subspace. This time, $\vec{\mathbf{0}}_3$ is in W, but W is not closed under addition. Produce an example of two vectors from W, but their sum is not in W. Note that W is also closed under scalar multiplication.

e. W is not a subspace. It contains $\vec{\mathbf{0}}_3$ and is closed under addition, but not under scalar multiplication. Remember, k can be any ***real*** number. As soon as k is ***irrational***, $k\vec{v}$ is not in W, if the coordinates of $\vec{v}$ are both integers.

f. $\{\langle 5, 0, 1, 0\rangle, \langle 0,-1, 0, 1\rangle\}$ is a basis for W, and $dim(W) = 2$.

g. $\{\langle -5,-5, 1, 0, 0\rangle, \langle 6, 6, 0, 1, 0\rangle, \langle -7, 0, 0, 0, 1\rangle\}$ is a basis for W, and $dim(W) = 3$.

h. $\{\langle 10, 10, 5, 2, 0\rangle, \langle 0,-1, 0, 0, 1\rangle\}$ is a basis for W, and $dim(W) = 2$.

i. W does not contain $\vec{\mathbf{0}}_4$.

j. W is not a subspace. Although W contains $\vec{\mathbf{0}}_4$, it is not closed under addition or scalar multiplication.

9. a. True b. False c. True d. False e. False

f. True g. True h. False i. False j. False.

Chapter Three Exercises

3.1 Exercises

1. a. f is a function since every parent has a unique oldest child. b. g is not a function because x may not have any daughter at all. c. h is a function because every person has a unique mother. d. k is not a function because y may not have any brother at all. e. p is not a function because even though x has at least one child, none of the children of x may have any children of their own. f. q is a function because the father of y is unique, say call him z, and the mother of z is also unique.

2. a. $\langle -15, 38, 5\rangle$. c. $[T] = \begin{bmatrix} 2 & 3 \\ 1 & -5 \\ 4 & 1 \end{bmatrix}$.

3. a. $\langle -25, -6, -9\rangle$. c. $[T] = \begin{bmatrix} 2 & 0 & -5 & 0 \\ 0 & 3 & 1 & -2 \\ 3 & 8 & 0 & 0 \end{bmatrix}$.

4. a. $\langle 55, -21, 58, 84\rangle$. c. $[T] = \begin{bmatrix} 3 & 2 & -5 \\ 1 & 0 & 4 \\ 0 & 2 & -7 \\ 4 & 9 & 0 \end{bmatrix}$.

5. a. $\langle 23, 62, -10\rangle$. c. $[T] = \begin{bmatrix} 5 & -3 & -2 \\ 4 & -6 & 3 \\ 2 & 2 & 0 \end{bmatrix}$.

6. No. T is neither additive nor homogeneous.

7. No. T is neither additive nor homogeneous.

8. a. $[T] = \begin{bmatrix} 0 & 2 \\ -5 & 4 \\ 3 & -7 \end{bmatrix}$. b. $\langle -4, -43, 35\rangle$ c. $T(\langle x, y\rangle) = \langle 2y, -5x + 4y, 3x - 7y\rangle$.

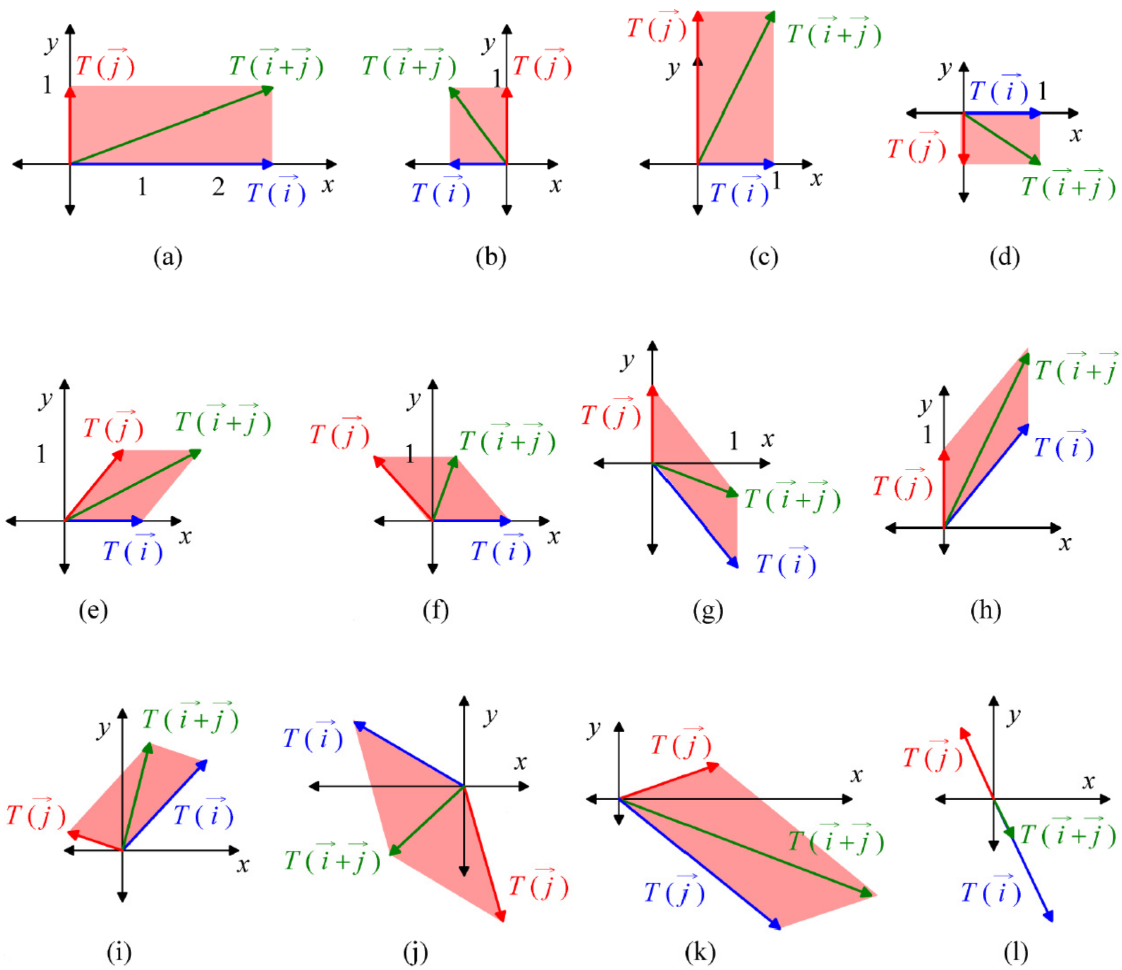
9. a. $[T] = \begin{bmatrix} -3 & 2 & 0 \\ 5 & 7 & 4 \end{bmatrix}$. b. $T(\langle x, y, z\rangle) = \langle -3x + 2y, 5x + 7y + 4z\rangle$ c. $\langle -19, 35\rangle$.

10. a. $[T] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

b. $T(\langle x_1, x_2, x_3, x_4, x_5 \rangle) = \langle x_5, x_3, x_1, x_4, x_2 \rangle$ c. $\langle 9, -5, 3, 2, 0 \rangle$.

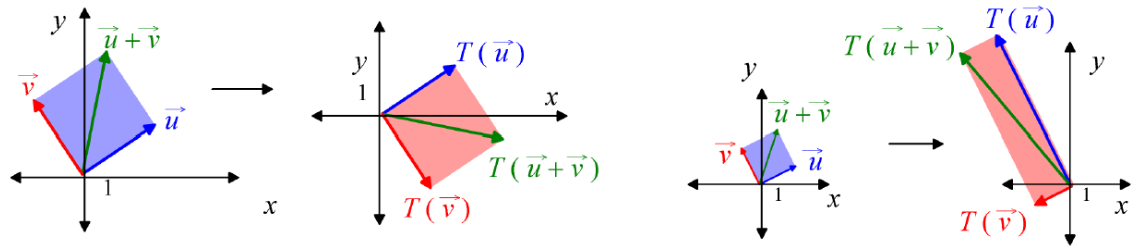
11. $T(\vec{v}_1) = \langle 6, -4, 17 \rangle$ and $T(\vec{v}_2) = \langle -13, 10, -44 \rangle$.

12. Answers:



The box in (l) “collapsed” into a line, because the two columns are parallel.

13. Answers:



(a) (b)

14. a. Yes, Type 3. b. No. c. No. d. Yes, Type 2. e. No. f. No. g. Yes, Type 1. h. No. i. No. j. No. k. No. l. Yes, Type 2.

16. $[S_k] = \begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{bmatrix}$

3.2 Exercises

1. ***Rotation Matrices:***

a. $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$; $rot_\theta(\langle 5,3 \rangle) = \left\langle \frac{5\sqrt{3}-3}{2}, \frac{3\sqrt{3}+5}{2} \right\rangle$

b. $\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$; $rot_\theta(\langle 5,3 \rangle) = \left\langle -\frac{3\sqrt{3}+5}{2}, \frac{5\sqrt{3}-3}{2} \right\rangle$

c. $\begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$; $rot_\theta(\langle 5,3 \rangle) = \langle 11/5, 27/5 \rangle$

d. $\begin{bmatrix} -5/13 & -12/13 \\ 12/13 & -5/13 \end{bmatrix}$; $rot_\theta(\langle 5,3 \rangle) = \langle -61/13, 45/13 \rangle$

e. $\begin{bmatrix} 12/13 & -5/13 \\ 5/13 & 12/13 \end{bmatrix}$; $rot_\theta(\langle 5,3 \rangle) = \langle 45/13, 61/13 \rangle$

f. $\begin{bmatrix} -\frac{1}{2}\sqrt{2-\sqrt{2}} & -\frac{1}{2}\sqrt{\sqrt{2}+2} \\ \frac{1}{2}\sqrt{\sqrt{2}+2} & -\frac{1}{2}\sqrt{2-\sqrt{2}} \end{bmatrix}$;

$$rot_\theta(\langle 5,3 \rangle) = \left\langle -\tfrac{3}{2}\sqrt{\sqrt{2}+2} - \tfrac{5}{2}\sqrt{-\sqrt{2}+2}, \tfrac{5}{2}\sqrt{\sqrt{2}+2} - \tfrac{3}{2}\sqrt{-\sqrt{2}+2} \right\rangle$$

$$\approx \langle -4.685, 3.471 \rangle$$

2. **Clockwise Rotations:**

a. $\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \left\langle \dfrac{-5+3\sqrt{3}}{2}, \dfrac{-3-5\sqrt{3}}{2} \right\rangle$

b. $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \left\langle -\dfrac{3-5\sqrt{3}}{2}, -\dfrac{5+3\sqrt{3}}{2} \right\rangle$

c. $\begin{bmatrix} 21/29 & 20/29 \\ -20/29 & 21/29 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \langle 165/29, -37/29\rangle$

d. $\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \langle 27/5, -11/5\rangle$

e. $\begin{bmatrix} -8/17 & 15/17 \\ -15/17 & -8/17 \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \langle 5/17, -99/17\rangle$

f. $\begin{bmatrix} -\dfrac{41}{841} & \dfrac{840}{841} \\ -\dfrac{840}{841} & -\dfrac{41}{841} \end{bmatrix}$; $rot_\theta(\langle 5,3\rangle) = \left\langle \dfrac{2315}{841}, -\dfrac{4323}{841} \right\rangle \approx \langle 2.75, -5.14\rangle$

3. **Projections and Reflections in $\mathbb{R}^2$:**

a. $[proj_L] = \begin{bmatrix} 25/34 & 15/34 \\ 15/34 & 9/34 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 9/34 & -15/34 \\ -15/34 & 25/34 \end{bmatrix}$;

$[refl_L] = \begin{bmatrix} 8/17 & 15/17 \\ 15/17 & -8/17 \end{bmatrix}$;

$proj_L(\langle 3,2\rangle) = \langle 105/34, 63/34\rangle$;
$proj_{L^\perp}(\langle 3,2\rangle) = \langle -3/34, 5/34\rangle$; $refl_L(\langle 3,2\rangle) = \langle 54/17, 29/17\rangle$

b. $[proj_L] = \begin{bmatrix} 49/65 & 28/65 \\ 28/65 & 16/65 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 16/65 & -28/65 \\ -28/65 & 49/65 \end{bmatrix}$;

$[refl_L] = \begin{bmatrix} 33/65 & 56/65 \\ 56/65 & -33/65 \end{bmatrix}$;

$proj_L(\langle 3,2\rangle) = \langle 203/65, 116/65\rangle$;
$proj_{L^\perp}(\langle 3,2\rangle) = \langle -8/65, 14/65\rangle$; $refl_L(\langle 3,2\rangle) = \langle 211/65, 102/65\rangle$

c. $[proj_L] = \begin{bmatrix} 25/41 & -20/41 \\ -20/41 & 16/41 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 16/41 & 20/41 \\ 20/41 & 25/41 \end{bmatrix}$;

$[refl_L] = \begin{bmatrix} 9/41 & -40/41 \\ -40/41 & -9/41 \end{bmatrix}$;

$proj_L(\langle 3,2\rangle) = \langle 35/41, -28/41\rangle$;

$proj_{L^\perp}(\langle 3,2\rangle) = \langle 88/41, 110/41\rangle$; $refl_L(\langle 3,2\rangle) = \langle -53/41, -138/41\rangle$

d. $[proj_L] = \begin{bmatrix} 9/58 & -21/58 \\ -21/58 & 49/58 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 49/58 & 21/58 \\ 21/58 & 9/58 \end{bmatrix}$;
$[refl_L] = \begin{bmatrix} -20/29 & -21/29 \\ -21/29 & 20/29 \end{bmatrix}$;
$proj_L(\langle 3,2\rangle) = \langle -15/58, 35/58\rangle$;
$proj_{L^\perp}(\langle 3,2\rangle) = \langle 189/58, 81/58\rangle$; $refl_L(\langle 3,2\rangle) = \langle -102/29, -23/29\rangle$

e. $[proj_L] = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$;
$[refl_L] = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$;
$proj_L(\langle 3,2\rangle) = \langle 9/10, 27/10\rangle$;
$proj_{L^\perp}(\langle 3,2\rangle) = \langle 21/10, -7/10\rangle$; $refl_L(\langle 3,2\rangle) = \langle -6/5, 17/5\rangle$

f. $[proj_L] = \begin{bmatrix} 3/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 1/4 \end{bmatrix}$; $[proj_{L^\perp}] = \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix}$;
$[refl_L] = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$
$proj_L(\langle 3,2\rangle) = \left\langle \left(9-2\sqrt{3}\right)/4, \left(2-3\sqrt{3}\right)/4 \right\rangle$;
$proj_{L^\perp}(\langle 3,2\rangle) = \left\langle \left(3+2\sqrt{3}\right)/4, \left(6+3\sqrt{3}\right)/4 \right\rangle$;
$refl_L(\langle 3,2\rangle) = \left\langle \left(3-2\sqrt{3}\right)/2, -\left(2-3\sqrt{3}\right)/2 \right\rangle$

4. ***Projections and Reflections in*** $\mathbb{R}^3$***:***

a. $[proj_L] = \begin{bmatrix} 16/29 & 8/29 & -12/29 \\ 8/29 & 4/29 & -6/29 \\ -12/29 & -6/29 & 9/29 \end{bmatrix}$; $[proj_\Pi] = \begin{bmatrix} 13/29 & -8/29 & 12/29 \\ -8/29 & 25/29 & 6/29 \\ 12/29 & 6/29 & 20/29 \end{bmatrix}$;
$[refl_\Pi] = \begin{bmatrix} -3/29 & -16/29 & 24/29 \\ -16/29 & 21/29 & 12/29 \\ 24/29 & 12/29 & 11/29 \end{bmatrix}$;
$proj_L(\langle -5,4,7\rangle) = \langle -132/29, -66/29, 99/29\rangle$;
$proj_\Pi(\langle -5,4,7\rangle) = \langle -13/29, 182/29, 104/29\rangle$;
$refl_\Pi(\langle -5,4,7\rangle) = \langle 119/29, 248/29, 5/29\rangle$

b. $[proj_L] = \begin{bmatrix} 4/65 & -10/65 & 12/65 \\ -10/65 & 25/65 & -30/65 \\ 12/65 & -30/65 & 36/65 \end{bmatrix}$;

$$[proj_\Pi] = \begin{bmatrix} 61/65 & 10/65 & -12/65 \\ 10/65 & 40/65 & 30/65 \\ -12/65 & 30/65 & 29/65 \end{bmatrix};$$

$$[refl_\Pi] = \begin{bmatrix} 57/65 & 20/65 & -24/65 \\ 20/65 & 15/65 & 60/65 \\ -24/65 & 60/65 & -7/65 \end{bmatrix};$$

$proj_L(\langle -5,4,7\rangle) = \langle 24/65, -60/65, 72/65\rangle;$
$proj_\Pi(\langle -5,4,7\rangle) = \langle -349/65, 320/65, 383/65\rangle;$
$refl_\Pi(\langle -5,4,7\rangle) = \langle -373/65, 380/65, 311/65\rangle$

c. $$[proj_L] = \begin{bmatrix} 49/90 & -28/90 & -35/90 \\ -28/90 & 16/90 & 20/90 \\ -35/90 & 20/90 & 25/90 \end{bmatrix};$$

$$[proj_\Pi] = \begin{bmatrix} 41/90 & 28/90 & 35/90 \\ 28/90 & 74/90 & -20/90 \\ 35/90 & -20/90 & 65/90 \end{bmatrix};$$

$$[refl_\Pi] = \begin{bmatrix} -4/45 & 28/45 & 35/45 \\ 28/45 & 29/45 & -20/45 \\ 35/45 & -20/45 & 20/45 \end{bmatrix};$$

$proj_L(\langle -5,4,7\rangle) = \langle -301/45, 172/45, 215/45\rangle;$
$proj_\Pi(\langle -5,4,7\rangle) = \langle 76/45, 8/45, 100/45\rangle;$
$refl_\Pi(\langle -5,4,7\rangle) = \langle 377/45, -164/45, -115/45\rangle$

d. $$[proj_L] = \begin{bmatrix} 9/34 & 0 & 15/34 \\ 0 & 0 & 0 \\ 15/34 & 0 & 25/34 \end{bmatrix}; \ [proj_\Pi] = \begin{bmatrix} 25/34 & 0 & -15/34 \\ 0 & 1 & 0 \\ -15/34 & 0 & 9/34 \end{bmatrix};$$

$$[refl_\Pi] = \begin{bmatrix} 8/17 & 0 & -15/17 \\ 0 & 1 & 0 \\ -15/17 & 0 & -8/17 \end{bmatrix};$$

$proj_L(\langle -5,4,7\rangle) = \langle 30/17, 0, 50/17\rangle;$
$proj_\Pi(\langle -5,4,7\rangle) = \langle -115/17, 68/17, 69/17\rangle;$
$refl_\Pi(\langle -5,4,7\rangle) = \langle -145/17, 68/17, 19/17\rangle$

e. $$[proj_L] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4/53 & -14/53 \\ 0 & -14/53 & 49/53 \end{bmatrix}; \ [proj_\Pi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 49/53 & 14/53 \\ 0 & 14/53 & 4/53 \end{bmatrix};$$

$$[refl_\Pi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 45/53 & 28/53 \\ 0 & 28/53 & -45/53 \end{bmatrix}$$

$proj_L(\langle -5,4,7\rangle) = \langle 0,-82/53,287/53\rangle$;
$proj_\Pi(\langle -5,4,7\rangle) = \langle -265/53,294/53,84/53\rangle$;
$refl_\Pi(\langle -5,4,7\rangle) = \langle -265/53,376/53,-203/53\rangle$

f. $[proj_L] = \begin{bmatrix} 16/65 & -28/65 & 0 \\ -28/65 & 49/65 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $[proj_\Pi] = \begin{bmatrix} 49/65 & 28/65 & 0 \\ 28/65 & 16/65 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$$[refl_\Pi] = \begin{bmatrix} 33/65 & 56/65 & 0 \\ 56/65 & -33/65 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$proj_L(\langle -5,4,7\rangle) = \langle -192/65,336/65,0\rangle$;
$proj_\Pi(\langle -5,4,7\rangle) = \langle -133/65,-76/65,7\rangle$;
$refl_\Pi(\langle -5,4,7\rangle) = \langle 59/65,-412/65,7\rangle$

5. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; No, because of the -1.

6. $[refl_L] = \begin{bmatrix} -10/19 & -15/19 & 6/19 \\ -15/19 & 6/19 & -10/19 \\ 6/19 & -10/19 & -15/19 \end{bmatrix} = -[refl_\Pi]$.

7. $[refl_L] = \begin{bmatrix} -57/65 & -20/65 & 24/65 \\ -20/65 & -15/65 & -60/65 \\ 24/65 & -60/65 & 7/65 \end{bmatrix}$

8. a. $T(\vec{v}) = \langle 2,5\rangle$ and $T(\vec{w}) = \langle 4,-3\rangle$.
   c. it corresponds to $refl_L$
   e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is the matrix of the reflection across $y = z$, and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is the matrix of the reflection across $x = z$.
   f. $T(\langle x_1,x_2,x_3,x_4\rangle) = \langle x_1,x_4,x_3,x_2\rangle$; T exchanges the 2nd and 4th components of $\vec{v}$.
11. $6x - 3y + 8z = 0$.
12. a. $\sqrt{29}/\sqrt{38}$, $\sqrt{13}/\sqrt{38}$, $\sqrt{34}/\sqrt{38}$. The radicand in the numerator is the respective diagonal entry.

b. $\frac{15}{38}$; $\frac{-6}{38}$; $\frac{10}{38}$; c. $\cos(\alpha_{i,j}) = \frac{15}{\sqrt{377}}$; $\alpha_{i,j} = \cos^{-1}\left(\frac{15}{\sqrt{377}}\right) \approx 39.42^0$

$\cos(\alpha_{i,k}) = \frac{-6}{\sqrt{986}}$; $\alpha_{i,k} = \cos^{-1}\left(\frac{-6}{\sqrt{986}}\right) \approx 101.02^0$; $\cos(\alpha_{j,k}) = \frac{10}{\sqrt{442}}$;

$\alpha_{j,k} = \cos^{-1}\left(\frac{10}{\sqrt{442}}\right) \approx 61.60^0$

3.3 Exercises

1. Answers:

a. $(T_1 + T_2)(\langle x, y, z\rangle) = \langle 5x - 2y + 14z, 2x + 3y - 4z\rangle$.

b. $[T_1 + T_2] = \begin{bmatrix} 5 & -2 & 14 \\ 2 & 3 & -4 \end{bmatrix}$

c. $[T_1] = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 4 & -7 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 2 & 0 & 9 \\ 1 & -1 & 3 \end{bmatrix}$

d. Yes!

e. $[-4T_1] = \begin{bmatrix} -12 & 8 & -20 \\ -4 & -16 & 28 \end{bmatrix} = -4[T_1]$.

2. Answers:

a. $(T_1 + T_2)(\langle x, y, z\rangle) = \langle 3x - 2y + 4z, 2x - y - 4z, x + 2y + 3z, -3x - y + z\rangle$.

b. $\begin{bmatrix} 3 & -2 & 4 \\ 2 & -1 & -4 \\ 1 & 2 & 3 \\ -3 & -1 & 1 \end{bmatrix}$

c. $[T_1] = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & -4 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 3 \\ -4 & 0 & 0 \end{bmatrix}$

d. Yes!

e. $\begin{bmatrix} 3 & -6 & 9 \\ 3 & 0 & -12 \\ 0 & 6 & 0 \\ 3 & -3 & 3 \end{bmatrix}$

3. Answers:

a. $\begin{bmatrix} -2 & -4 & -3 \\ 6 & 7 & -5 \end{bmatrix}$; 2×3

b. does not exist

c. $\begin{bmatrix} -3 & -11 \\ 4 & 26 \\ -29 & -15 \end{bmatrix}$; 3×2

d. does not exist

e. $\begin{bmatrix} -32 & 22 \\ 43 & -19 \\ -4 & -7 \end{bmatrix}$; 3×2

f. does not exist

g. $\begin{bmatrix} 57 & -40 \\ -20 & 17 \end{bmatrix}$; 2×2

h. does not exist

i. $\begin{bmatrix} 3 & 31 & -13 \\ -2 & -46 & 20 \\ 17 & -27 & 17 \end{bmatrix}$; 3×3

j. $\begin{bmatrix} 55 & 34 \\ -19 & 15 \end{bmatrix}$; 2×2

k. $\begin{bmatrix} 317 & -118 \\ -163 & 121 \end{bmatrix}$; 2×2

l. same as (k).

m. $\begin{bmatrix} -13 & 195 & -91 \\ 26 & -314 & 148 \\ 65 & -367 & 183 \end{bmatrix}$; 3×3

n. same as (m).

o. $\begin{bmatrix} 461 & 178 \\ -167 & -23 \end{bmatrix}$; 2×2

4. Answers:

a. $\begin{bmatrix} 1 & 8 & -15 \\ 37 & -52 & -69 \\ -28 & -17 & 2 \end{bmatrix}$; 3×3

b. $\begin{bmatrix} 56 & 5 & -35 & 55 \\ -1 & -29 & 18 & 16 \\ -3 & -24 & -13 & 39 \\ -39 & 4 & 41 & -63 \end{bmatrix}$; 4×4

c. $\begin{bmatrix} 5 & -15 & -15 & 13 & 70 \\ 93 & -35 & 63 & -49 & 88 \\ -63 & -15 & -14 & 31 & -4 \end{bmatrix}$; 3×5

d. does not exist

e. $\begin{bmatrix} 13 & -56 & 72 \\ 52 & -31 & -41 \\ -63 & 50 & 10 \\ 37 & -29 & -60 \end{bmatrix}$; 4×3

f. $\begin{bmatrix} -50 & -53 & 65 & -25 \\ 23 & 1 & 0 & 10 \\ 64 & 26 & -20 & 20 \\ -11 & -17 & -12 & 26 \\ -16 & -6 & 20 & -20 \end{bmatrix}$; 5×4

g. $\begin{bmatrix} 41 & -51 & -84 \\ -19 & 20 & 41 \\ 14 & -17 & -60 \\ 12 & 2 & 31 \\ 41 & 36 & -9 \end{bmatrix}$; 5×3

h. does not exist.

i. $\begin{bmatrix} 89 & 59 & -59 & 30 \\ -17 & -49 & -21 & 0 \\ -85 & 27 & 139 & -58 \\ 71 & 6 & -75 & -4 \end{bmatrix}$; 4×4

j. does not exist.

k. $\begin{bmatrix} 631 & -225 & 362 & -299 & 672 \\ -237 & -105 & -101 & 163 & 194 \\ 14 & -250 & 247 & -54 & 272 \\ -550 & 310 & -477 & 312 & -622 \end{bmatrix}$; 4×5

l. same as (k).

m. $\begin{bmatrix} 503 & -1 & -356 \\ -139 & -207 & -326 \\ -425 & 649 & 1340 \\ 306 & 83 & -560 \end{bmatrix}$; 4×3

n. same as (m).

o. $\begin{bmatrix} 717 & -153 & -597 \\ 45 & 4173 & 2895 \\ -713 & 626 & 1597 \end{bmatrix}$; 3×3

5. Answers:

a. The codomain of T_1 is $\mathbb{R}^4$, which is also the domain of T_2. The domain of $T_2 \circ T_1$ is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$

b. This composition is not well defined.

c. $\langle 9x - 26y, 33x + 9y, -6x + 54y \rangle$

d. $\begin{bmatrix} 9 & -26 \\ 33 & 9 \\ -6 & 54 \end{bmatrix}$

e. $[T_2] = \begin{bmatrix} 3 & 0 & 0 & -5 \\ 0 & 7 & 2 & -1 \\ 0 & 0 & 6 & 9 \end{bmatrix}$; $[T_1] = \begin{bmatrix} 3 & -2 \\ 5 & 1 \\ -1 & 3 \\ 0 & 4 \end{bmatrix}$;

$$[T_2][T_1] = \begin{bmatrix} 9 & -26 \\ 33 & 9 \\ -6 & 54 \end{bmatrix} = [T_2 \circ T_1].$$

6. Answers:

a. The codomain of one is the domain of the other, so both compositions are well-defined. $T_2 \circ T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_1 \circ T_2 : \mathbb{R}^4 \to \mathbb{R}^4$.

b. $(T_2 \circ T_1)(\langle x,y,z \rangle) = \langle 9x + 10y + 7z, 16x - 8y + 32z, 6x + 9y - 12z \rangle$, and

$$T_1 \circ T_2(\langle x_1,x_2,x_3,x_4 \rangle) = \langle 9x_1 + 35x_2 + 4x_3 - 29x_4, 6x_1 - 7x_2 + 22x_3 + 27x_4, 3x_1 + 6x_3 + 4x_4, 7x_2 - 10x_3 - 19x_4 \rangle$$

c. $[T_2 \circ T_1] = \begin{bmatrix} 9 & 10 & 7 \\ 16 & -8 & 32 \\ 6 & 9 & -12 \end{bmatrix}$, $[T_1 \circ T_2] = \begin{bmatrix} 9 & 35 & 4 & -29 \\ 6 & -7 & 22 & 27 \\ 3 & 0 & 6 & 4 \\ 0 & 7 & -10 & -19 \end{bmatrix}$

d. $[T_2] = \begin{bmatrix} 3 & 0 & 0 & -5 \\ 0 & 7 & 2 & -1 \\ 0 & 0 & 6 & 9 \end{bmatrix}$; $[T_1] = \begin{bmatrix} 3 & 5 & -1 \\ 2 & -1 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$;

$$[T_2][T_1] = \begin{bmatrix} 9 & 10 & 7 \\ 16 & -8 & 32 \\ 6 & 9 & -12 \end{bmatrix} = [T_2 \circ T_1];$$

$$[T_1][T_2] = \begin{bmatrix} 9 & 35 & 4 & -29 \\ 6 & -7 & 22 & 27 \\ 3 & 0 & 6 & 4 \\ 0 & 7 & -10 & -19 \end{bmatrix} = [T_1 \circ T_2]$$

7. Answers:

a. The codomain of one is the domain of the other, so both compositions are well-defined. $T_2 \circ T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_1 \circ T_2 : \mathbb{R}^5 \to \mathbb{R}^5$.

b. $(T_2 \circ T_1)(\langle x,y \rangle) = \langle 10x - 13y, 17x + 26y \rangle$, and
$(T_1 \circ T_2)(\langle x_1,x_2,x_3,x_4,x_5 \rangle) = \langle 21x_1 + 7x_2 - 2x_3 + 3x_4 - 6x_5,$
$21x_2 - 20x_3 + 16x_4 - 25x_5, 78x_1 + 35x_2 - 16x_3 + 18x_4 - 33x_5,$
$54x_1 + 12x_3 - 6x_4 + 6x_5, -6x_1 - 14x_2 + 12x_3 - 10x_4 + 16x_5 \rangle$

c. $[T_2 \circ T_1] = \begin{bmatrix} 10 & -13 \\ 17 & 26 \end{bmatrix}$; $[T_1 \circ T_2] = \begin{bmatrix} 21 & 7 & -2 & 3 & -6 \\ 0 & 21 & -20 & 16 & -25 \\ 78 & 35 & -16 & 18 & -33 \\ 54 & 0 & 12 & -6 & 6 \\ -6 & -14 & 12 & -10 & 16 \end{bmatrix}$

d. $[T_2] = \begin{bmatrix} 3 & 7 & -6 & 5 & -8 \\ 9 & 0 & 2 & -1 & 1 \end{bmatrix}$; $[T_1] = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 5 & 7 \\ 0 & 6 \\ -2 & 0 \end{bmatrix}$;

$$[T_2][T_1] = \begin{bmatrix} 10 & -13 \\ 17 & 26 \end{bmatrix} = [T_2 \circ T_1];$$

$$[T_1][T_2] = \begin{bmatrix} 21 & 7 & -2 & 3 & -6 \\ 0 & 21 & -20 & 16 & -25 \\ 78 & 35 & -16 & 18 & -33 \\ 54 & 0 & 12 & -6 & 6 \\ -6 & -14 & 12 & -10 & 16 \end{bmatrix} = [T_1 \circ T_2]$$

11. If A is $m \times k$, then B has to be $k \times m$. For both compositions to be defined, m must equal n.

3.4 Exercises

1. a. $\begin{bmatrix} 11 & -7 & -1 & 11 \\ -6 & 4 & 0 & 1 \\ -7 & 13 & 8 & 4 \end{bmatrix}$ b. $\begin{bmatrix} 96 & 138 \\ -54 & -32 \\ -72 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 56 & 75 \\ 0 & 77 \\ -40 & -15 \end{bmatrix}$

   d. $\begin{bmatrix} 40 & 63 \\ -54 & -109 \\ -32 & 20 \end{bmatrix}$ e. $\begin{bmatrix} 96 & 138 \\ -54 & -32 \\ -72 & 5 \end{bmatrix}$ f. $\begin{bmatrix} 2 & 12 \\ -3 & -2 \\ 0 & 6 \\ 8 & 7 \end{bmatrix}$

   g. $\begin{bmatrix} 48 & 59 \\ -64 & -132 \\ 5 & 8 \end{bmatrix}$ h. $\begin{bmatrix} 8 & -4 \\ -10 & -23 \\ 37 & -12 \end{bmatrix}$

   i. $\begin{bmatrix} 48 & 59 \\ -64 & -132 \\ 5 & 8 \end{bmatrix}$ j. $\begin{bmatrix} 131 & 217 \\ -16 & -73 \\ -21 & -34 \end{bmatrix}$

2. Answers:

   a. $[T_1] = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 5 & -7 \\ 1 & -1 & 4 \\ 6 & 1 & -1 \end{bmatrix}$; 4×3; $[T_2] = \begin{bmatrix} 5 & 0 & 2 & -1 \\ 2 & 8 & -6 & 7 \end{bmatrix}$; 2×4;

$$[T_3] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 7 & 3 \\ 4 & 1 \\ 1 & 5 \end{bmatrix}; \quad 5 \times 2$$

b. $(T_2 \circ T_1)(\langle x,y,z \rangle) = \langle 6x - 18y + 9z, 40x + 47y - 87z \rangle$.

c. $\begin{bmatrix} 6 & -18 & 9 \\ 40 & 47 & -87 \end{bmatrix}; \ 2 \times 3$

d. same as c.

e. $\begin{bmatrix} 9 & 16 & -10 & 13 \\ 3 & -8 & 8 & -8 \\ 41 & 24 & -4 & 14 \\ 22 & 8 & 2 & 3 \\ 15 & 40 & -28 & 34 \end{bmatrix} (5 \times 4)$; f. $\begin{bmatrix} 86 & 76 & -165 \\ -34 & -65 & 96 \\ 162 & 15 & -198 \\ 64 & -25 & -51 \\ 206 & 217 & -426 \end{bmatrix} (5 \times 3)$.

3. Answers:

a. $[T_1] = \begin{bmatrix} 8/17 & 15/17 \\ 15/17 & -8/17 \end{bmatrix}; \ [T_2] = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix};$

$[T_3] = \begin{bmatrix} 9/58 & -21/58 \\ -21/58 & 49/58 \end{bmatrix}.$

b. $[T_2 \circ T_1] = \begin{bmatrix} \frac{84}{85} & \frac{13}{85} \\ \frac{13}{85} & -\frac{84}{85} \end{bmatrix}; \ [T_1 \circ T_3] = \begin{bmatrix} -\frac{243}{986} & \frac{567}{986} \\ \frac{303}{986} & -\frac{707}{986} \end{bmatrix}$

c. $[T_3 \circ T_2 \circ T_1] = \begin{bmatrix} \frac{483}{4930} & \frac{1881}{4930} \\ -\frac{1127}{4930} & -\frac{4389}{4930} \end{bmatrix}; \ [T_1 \circ T_3 \circ T_2] = \begin{bmatrix} -\frac{2997}{4930} & \frac{729}{4930} \\ \frac{3737}{4930} & -\frac{909}{4930} \end{bmatrix};$

we get different answers.

4. $[T_1] = \begin{bmatrix} 2 & -3 & 1 \\ 4 & -5 & -7 \end{bmatrix} (2 \times 3), \ [T_2] = \begin{bmatrix} 5 & -4 \\ 1 & -3 \\ 7 & 2 \end{bmatrix} (3 \times 2),$

$[T_1 \circ T_2] = \begin{bmatrix} 14 & 3 \\ -34 & -15 \end{bmatrix} (2 \times 2), \ [T_2 \circ T_1] = \begin{bmatrix} -6 & 5 & 33 \\ -10 & 12 & 22 \\ 22 & -31 & -7 \end{bmatrix} (3 \times 3).$

5. $A^2 = \begin{bmatrix} 44 & -35 \\ -25 & 39 \end{bmatrix}$, $A^3 = \begin{bmatrix} -307 & 378 \\ 270 & -253 \end{bmatrix}$, $A^4 = \begin{bmatrix} 2811 & -2905 \\ -2075 & 2396 \end{bmatrix}$.

$p(A) = 4I_2 - 6A + 5A^2 - 2A^3 + 7A^4 = \begin{bmatrix} 20,533 & -21,308 \\ -15,220 & 17,489 \end{bmatrix}$.

Reminder: the first term is $4I_2$.

6. $A^2 = \begin{bmatrix} 3 & -8 & -16 \\ 0 & 1 & -6 \\ 4 & 24 & 51 \end{bmatrix}$, $A^3 = \begin{bmatrix} -5 & -56 & -118 \\ 9 & -25 & -42 \\ 25 & 180 & 349 \end{bmatrix}$, $p(A) = \begin{bmatrix} -15 & -72 & -170 \\ 39 & -59 & -54 \\ 23 & 268 & 495 \end{bmatrix}$

7. a. We have two non-zero, non-parallel vectors. b. $\langle 217, 579, -694 \rangle$

8. a. The rref of the matrix with the 3 vectors as columns is I_3. b. $\langle 18, \frac{19}{2}, \frac{57}{2}, -7, \frac{59}{2} \rangle$

12. a. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$;

rotate $\mathbb{R}^2$ by θ, then reflect $\mathbb{R}^2$ across the y-axis.

b. $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$;

reflect $\mathbb{R}^2$ across the x-axis, then rotate $\mathbb{R}^2$ by θ.

15. Rotating $\mathbb{R}^2$ by α, followed by another rotation by β results in a net rotation by $\alpha + \beta$. Similarly, rotating $\mathbb{R}^2$ by β, followed by another rotation by α results in a net rotation by $\beta + \alpha$, which is the same as $\alpha + \beta$.

3.5 Exercises

1. a. $[T_1] = \begin{bmatrix} 3 & 1 & -7 & 8 \\ 2 & 2 & -2 & -4 \\ -2 & 1 & 8 & -17 \end{bmatrix}$ b. $R_1 = \begin{bmatrix} 1 & 0 & -3 & 5 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\{\langle 3,-2,1,0 \rangle, \langle -5,7,0,1 \rangle\}$ d. $nullity(T_1) = 2$

e. T_1 is not 1-1. f. $\{\langle 3,2,-2 \rangle, \langle 1,2,1 \rangle\}$

g. $rank(T_1) = 2$ h. T_1 is not onto. i. $2 + 2 = 4$.

2. a. $[T_2] = \begin{bmatrix} 3 & -6 & 5 \\ 2 & -4 & 7 \\ -5 & 10 & 3 \\ -1 & 2 & 8 \end{bmatrix}$ b. $R_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c. $\{\langle 2,1,0 \rangle\}$ d. $nullity(T_2) = 1$ e. T_2 is not 1-1.

f. $\{\langle 3,2,-5,-1 \rangle, \langle 5,7,3,8 \rangle\}$ g. $rank(T_2) = 2$

h. T_2 is not onto. i. $2 + 1 = 3$.

3. a. $[T_3] = \begin{bmatrix} -5 & -7 & 2 \\ -2 & 1 & 16 \\ 3 & -2 & -26 \end{bmatrix}$ b. $R_3 = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

c. $\{\langle 6,-4,1\rangle\}$ d. $nullity(T_3) = 1$ e. T_3 is not 1-1.

f. $\{\langle -5,-2,3\rangle,\langle -7,1,-2\rangle\}$ g. $rank(T) = 2$

h. T_3 is not onto. i. $2+1 = 3$

j. The kernel is a line with direction $\langle 6,-4,1\rangle$, and the range is a plane with equation $x - 31y - 19z = 0$

k. The kernel is not necessarily orthogonal to the range (columnspace). The kernel is always orthogonal to the ***rowspace***.

4. Answers:

a. (i) $\{\langle -5,2,1\rangle\}$ (ii) 1 (iii) T is not one-to-one. (iv) $\{\langle 2,3,3,-3,3\rangle,\langle 3,4,5,2,10\rangle\}$ (v) 2; (vi) T is not onto. (vii) not full-rank. (viii) $2+1 = 3$.

b. (i) there is no basis for the kernel of T. (ii) 0 (iii) T is one-to-one. (iv) $\{\langle 2,3,3,-3,3\rangle,\langle 3,4,5,2,10\rangle,\langle 4,7,5,-18,-5\rangle\}$ (v) 3 (vi) T is not onto. (vii) full-rank. (viii) $3+0 = 3$.

c. (i) $\{\langle -4,-9,1,0,0\rangle,\langle 5,3,0,1,0\rangle,\langle -2,1,0,0,1\rangle\}$ (ii) 3 (iii) T is not one-to-one. (iv) d. $\{\langle 3,-5,-8\rangle,\langle -2,3,5\rangle\}$ (v) 2 (vi) T is not onto. (vii) not full-rank. (viii) $2+3 = 5$.

d. (i) $\{\langle -4,-9,1,0,0\rangle,\langle 5,3,0,1,0\rangle\}$ (ii) 2 (iii) T is not one-to-one. (iv) $\{\langle 3,-5,-8\rangle,\langle -2,3,5\rangle,\langle 8,-13,-20\rangle\}$ (v) 3 (vi) T is onto. (vii) full-rank. (viii) $3+2 = 5$.

e. (i) $\{\langle -2,-3,1,1,0\rangle,\langle 1,-2,5,0,1\rangle\}$ (ii) 2 (iii) T is not one-to-one. (iv) $\{\langle 3,-5,-8\rangle,\langle -2,3,5\rangle,\langle -2,9,4\rangle\}$ (v) 3 (vi) T is onto. (vii) full-rank. (viii) $3+2 = 5$.

f. (i) $\{\langle -4,-9,1,0,0\rangle,\langle -3,8,0,-5,1\rangle\}$ (ii) 2 (iii) T is not one-to-one. (iv) $\{\langle 3,-5,-8,6\rangle,\langle -2,3,5,-3\rangle,\langle -2,9,10,-8\rangle\}$ (v) 3 (vi) T is not onto. (vii) not full-rank. (viii) $3+2 = 5$.

g. (i) $\{\langle 3,1,0,0,0\rangle,\langle 7,0,-5,1,0\rangle\}$ (ii) 2 (iii) T is not one-to-one. (iv) $\{\langle 3,-5,-2,2\rangle,\langle 6,-7,-3,5\rangle,\langle -2,9,7,-8\rangle\}$ (v) 3 (vi) T is not onto. (vii) not full-rank. (viii) $3+2 = 5$.

h. (i) $\{\langle -2,1,-3,-5,1\rangle\}$ (ii) 1 (iii) T is not one-to-one. (iv) $\{\langle 3,-5,-2,2\rangle,\langle 6,-7,-3,5\rangle,\langle -2,3,4,7\rangle,\langle -1,-4,3,-2\rangle\}$ (v) 4 (vi) T is onto. (vii) full-rank. (viii) $4+1 = 5$.

i. (i) $\{\langle -5,2,1,0,0\rangle\}$ (ii) 1 (iii) T is not one-to-one. (iv) $\{\langle 3,-5,-2,2\rangle,\langle 6,-7,-3,5\rangle,\langle -2,3,4,7\rangle,\langle -1,-4,3,-2\rangle\}$ (v) 4 (vi) T is onto. (vii) full-rank. (viii) $4+1 = 5$.

j. (i) $\{\langle -72,25,45,0\rangle,\langle -36,35,0,45\rangle\}$ (ii) 2 (iii) T is not one-to-one. (iv) $\{\langle 15,30,-10,-5,-15\rangle,\langle 72,63,27,-54,0\rangle\}$ (v) 2 (vi) T is not onto. (vii) not full-rank. (viii) $2+2 = 4$.

k. (i) there is no basis for the kernel of T (ii) 0 (iii) T is one-to-one. (iv) $\{\langle 1,3,-1,-5,5\rangle,\langle 2,6,7,-4,0\rangle,\langle -6,3,-3,2,-4\rangle,\langle -4,-5,-2,3,1\rangle\}$

(v) 4 (vi) T is not onto. (vii) full-rank. (viii) $4+0 = 4$.

l. (i) $\{\langle -4,3,-2,1\rangle\}$ (ii) 1 (iii) T is not one-to-one.
(iv) $\{\langle 5,2,-6,-2,1\rangle,\langle 7,-1,-3,3,0\rangle,\langle 2,3,-5,1,-1\rangle\}$
(v) 3 (vi) T is not onto. (vii) not full-rank. (viii) $3+1 = 4$.

m. (i) $\{\langle 3,1,0,0\rangle,\langle -4,0,2,1\rangle\}$ (ii) 2 (iii) T is not one-to-one.
(iv) $\{\langle 2,3,2,5\rangle,\langle 3,1,5,4\rangle\}$ (v) 2 (vi) T is not onto. (vii) not full-rank.
(viii) $2+2 = 4$.

n. (i) $\{\langle 5,-3,-8,1\rangle\}$ (ii) 1 (iii) T is not one-to-one.
(iv) $\{\langle 4,5,-6,5\rangle,\langle 2,9,-7,6\rangle,\langle 1,-2,-1,3\rangle\}$
(v) 3 (vi) T is not onto. (vii) not full-rank. (viii) $3+1 = 4$.

o. (i) $\{\langle 5,1,0,0,0\rangle,\langle -9,0,7,1,0\rangle\}$ (ii) 2 (iii) T is not one-to-one.
(iv) $\{\langle -3,2,5,0,-4\rangle,\langle -5,-1,2,-3,-7\rangle,\langle 12,-4,0,-25,37\rangle\}$
(v) 3 (vi) T is not onto. (vii) not full-rank. (viii) $3+2 = 5$.

p. (i) $\{\langle 7,-5,1,0,0\rangle\}$ (ii) 1 (iii) T is not one-to-one.
(iv) $\{\langle -3,2,4,0,-3\rangle,\langle -5,-1,6,-1,-4\rangle,\langle 2,-4,-5,-5,3\rangle,\langle -5,-1,2,-3,-7\rangle\}$
(v) 4 (vi) T is not onto. (vii) not full-rank. (viii) $4+1 = 5$.

q. (i) $\{\langle -7,2,-3,1,0\rangle,\langle 5,-3,2,0,1\rangle\}$ (ii) 2 (iii) T is not one-to-one.
(iv) $\{\langle -3,2,4,0,-3\rangle,\langle -5,-1,2,-3,-7\rangle,\langle 2,-4,-5,-5,3\rangle\}$
(v) 3 (vi) T is not onto. (vii) not full-rank. (viii) $3+2 = 5$.

6. a. Π b. L c. L d. Π e. $\left\{\vec{\mathbf{0}}_3\right\}$ f. $\mathbb{R}^3$

7. The three image vectors are linearly dependent:
$\frac{8}{5}\langle 2,-3,4,-1,7\rangle - \frac{3}{5}\langle -3,2,-1,4,2\rangle = \langle 5,-6,7,-4,10\rangle$, so
$\frac{8}{5}\langle 1,-2,1\rangle - \frac{3}{5}\langle 0,-1,3\rangle - \langle 0,-2,5\rangle = \left\langle \frac{8}{5},-\frac{3}{5},-\frac{26}{5}\right\rangle$ is a non-zero vector in $ker(T)$.

16. a. True. b. False. c. True. d. False. e. False. f. True. g. True. h. True. i. False. j. True. k. False. l. False. m. True. n. True.

3.6 Exercises

1. Answers:

a. $\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

b. $\begin{bmatrix} \frac{1}{5} & \frac{7}{20} \\ 0 & -\frac{1}{4} \end{bmatrix}$

c. $\begin{bmatrix} 0 & \frac{1}{6} \\ \frac{1}{4} & 0 \end{bmatrix}$

d. $\begin{bmatrix} 4 & -9 \\ -3 & 7 \end{bmatrix}$

e. $\begin{bmatrix} -1 & -2 \\ -\frac{4}{3} & -\frac{7}{3} \end{bmatrix}$

f. $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & -\frac{3}{8} \end{bmatrix}$

g. $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

h. not invertible.

i. $\begin{bmatrix} \frac{11}{19} & -\frac{5}{57} \\ \frac{14}{19} & \frac{4}{57} \end{bmatrix}$

j. $\begin{bmatrix} -\frac{27}{124} & \frac{11}{124} \\ \frac{12}{31} & \frac{2}{31} \end{bmatrix}$

k. $\begin{bmatrix} \frac{105}{179} & -\frac{24}{179} \\ \frac{10}{179} & \frac{100}{179} \end{bmatrix}$

l. $\frac{1}{24}\begin{bmatrix} -\sqrt{6} & \sqrt{30} \\ 2\sqrt{15} & -2\sqrt{3} \end{bmatrix}$

2. ***Symbolic Matrices**:*

a. $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, which is the matrix of the ***clockwise*** rotation by θ.

b. $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

c. $\frac{1}{5}\begin{bmatrix} 3e^{-3x} & e^{2x} \\ -2e^{-4x} & e^{x} \end{bmatrix}$

d. $\dfrac{1}{2 \cdot 60^x}\begin{bmatrix} 10^x & 4^x \\ 15^x & -6^x \end{bmatrix}$

e. $\begin{bmatrix} \cosh(x) & -\sinh(x) \\ -\sinh(x) & \cosh(x) \end{bmatrix}$

f. not invertible.

g. $\begin{bmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{bmatrix}$

3. ***Inverses of Operators***:

a. $[T] = \begin{bmatrix} 3 & -7 \\ -4 & 9 \end{bmatrix}$; $[T]^{-1} = \begin{bmatrix} -9 & -7 \\ -4 & -3 \end{bmatrix}$;

$T^{-1}(\langle x,y\rangle) = \langle -9x - 7y, -4x - 3y\rangle$

b. T is not invertible.

c. $[T] = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$; $[T]^{-1} = \begin{bmatrix} \frac{9}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$;

$T^{-1}(\langle x,y\rangle) = \langle 9x/2 - 5y/2, -5x/2 + 3y/2\rangle$.

d. $[T] = \begin{bmatrix} \frac{2}{3} & \frac{5}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix}$; $[T]^{-1} = \begin{bmatrix} \frac{3}{22} & \frac{15}{22} \\ \frac{6}{11} & -\frac{3}{11} \end{bmatrix}$;

$T^{-1}(\langle x,y\rangle) = \langle 3x/22 + 15y/22, 6x/11 - 3y/11\rangle$.

5. a. $\begin{bmatrix} 6 & -21 \\ 10 & -35 \end{bmatrix}$; e. $\begin{bmatrix} 31 & -27 \\ -59 & 69 \end{bmatrix}$; Yes. f. $\begin{bmatrix} 31 & 124 \\ -93 & -372 \end{bmatrix}$; No. g. No.

3.7 Exercises

1. Note: answers vary for (ii) and (iii), so only answers to (i) are provided.

a. $\begin{bmatrix} 2 & -\frac{7}{3} \\ -1 & \frac{4}{3} \end{bmatrix}$ b. $\begin{bmatrix} \frac{7}{11} & -\frac{3}{11} \\ \frac{10}{11} & \frac{2}{11} \end{bmatrix}$ c. $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 1 \\ 1 & -1 & -1 \\ \frac{9}{5} & -\frac{12}{5} & -2 \end{bmatrix}$

d. not invertible. e. $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{11}{6} \\ 0 & -\frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ f. $\begin{bmatrix} 2 & 0 & 0 \\ \frac{3}{4} & \frac{3}{2} & 0 \\ \frac{13}{12} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$

g. $\begin{bmatrix} \frac{5}{31} & -\frac{2}{31} & -\frac{4}{31} \\ -\frac{1}{62} & \frac{19}{62} & \frac{7}{62} \\ \frac{8}{31} & \frac{3}{31} & \frac{6}{31} \end{bmatrix}$ h. $\begin{bmatrix} \frac{8}{27} & \frac{2}{27} & \frac{11}{27} \\ \frac{7}{27} & -\frac{5}{27} & \frac{13}{27} \\ -\frac{4}{9} & -\frac{1}{9} & -\frac{1}{9} \end{bmatrix}$ i. $\begin{bmatrix} \frac{3}{7} & \frac{1}{7} & \frac{4}{7} \\ 1 & -1 & 2 \\ -\frac{9}{7} & -\frac{3}{7} & \frac{2}{7} \end{bmatrix}$

j. $\begin{bmatrix} -1 & \frac{3}{2} & 11 & \frac{25}{6} \\ 0 & \frac{1}{2} & 2 & \frac{1}{6} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$ k. not invertible. l. $\begin{bmatrix} \frac{4}{7} & \frac{11}{14} & -\frac{13}{14} & -\frac{6}{7} \\ \frac{1}{7} & \frac{4}{7} & -\frac{6}{7} & -\frac{5}{7} \\ \frac{5}{7} & \frac{6}{7} & -\frac{9}{7} & -\frac{11}{7} \\ -\frac{2}{7} & -\frac{9}{14} & \frac{3}{14} & \frac{3}{7} \end{bmatrix}$

2. Answers:
   a. Multiply row 2 of A by −5.
   b. Multiply row 3 of A by −2/5.
   c. Add 3 times row A to row 2 of A.
   d. Add 7 times row 2 of A to row 3 of A.
   e. Exchange rows 1 and 3 of A.
   f. Subtract 4 times row 3 of A from row 1 of A.
3. Answers:
   a. Subtract 3 times row 4 of A from row 2 of A.
   b. Exchange rows 2 and 4 of A.
   c. Multiply row 3 of A by 3/2.
   d. Multiply row 4 of A by 9.
   e. Add 5 times row 2 of A to row 4 of A.
   f. Exchange rows 1 and 4 of A.
12. Answers:
   a. Subtract 3 times column 2 of A from column 4 of A.
   b. Exchange columns 2 and 4 of A.
   c. Multiply column 3 of A by 3/2.
   d. Multiply column 4 of A by 9.
   e. Add 5 times column 1 of A to column 3 of A.
   f. Exchange columns 1 and 4 of A.

3.8 Exercises

1. Answers will depend on the sequence of row operations you performed to get the rref.
2. Answers:

a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{55}{31} \\ -\frac{73}{62} \\ \frac{26}{31} \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{20}{9} \\ -\frac{31}{9} \\ -\frac{2}{3} \end{bmatrix}$

c. $\begin{bmatrix} u & x \\ v & y \\ w & z \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & -\frac{24}{7} \\ -4 & -18 \\ \frac{16}{7} & -\frac{26}{7} \end{bmatrix}$

d. $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{253}{6} \\ -\frac{37}{6} \\ -\frac{11}{3} \\ -\frac{1}{3} \end{bmatrix}$

e. $\begin{bmatrix} s & w \\ t & x \\ u & y \\ v & z \end{bmatrix} = \begin{bmatrix} -\frac{62}{7} & \frac{179}{14} \\ -\frac{61}{7} & \frac{60}{7} \\ -\frac{102}{7} & \frac{132}{7} \\ \frac{24}{7} & -\frac{121}{14} \end{bmatrix}$

3. a. $A^{-1} = \begin{bmatrix} -3 & \frac{5}{2} \\ 2 & -\frac{3}{2} \end{bmatrix}$; $B^{-1} = \begin{bmatrix} -1 & \frac{4}{3} \\ -2 & \frac{7}{3} \end{bmatrix}$ b. $\begin{bmatrix} \frac{17}{3} & -\frac{9}{2} \\ \frac{32}{3} & -\frac{17}{2} \end{bmatrix}$

c. $\begin{bmatrix} 51 & -27 \\ 64 & -34 \end{bmatrix}$ d. $\begin{bmatrix} \frac{17}{3} & -\frac{9}{2} \\ \frac{32}{3} & -\frac{17}{2} \end{bmatrix}$

4. $A^{-1} = BX^{-1}$ and $B^{-1} = X^{-1}A$.
7. B^{-1} is obtained from A^{-1} by exchanging columns 1 and 3 of A^{-1}, followed by exchanging columns 2 and 5.

8. ***Direct Sums and Matrices in Block Diagonal Form:***

a. $B = \begin{bmatrix} 5 & -2 & 1 & 0 & 0 \\ -4 & 0 & 7 & 0 & 0 \\ 3 & -9 & -8 & 0 & 0 \\ 0 & 0 & 0 & 3 & -7 \\ 0 & 0 & 0 & -2 & 4 \end{bmatrix}$; $C = \begin{bmatrix} 3 & -7 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 3 & -7 \\ 0 & 0 & -2 & 4 \end{bmatrix}$

b. The entries don't match because the matrices are in opposite locations.

c. $A_2 \oplus A_3 = \begin{bmatrix} 5 & -2 & 1 & 0 & 0 \\ -4 & 0 & 7 & 0 & 0 \\ 3 & -9 & -8 & 0 & 0 \\ 0 & 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & 7 & -3 \end{bmatrix}$;

$$(A_1 \oplus A_2) \oplus A_3 = A_1 \oplus (A_2 \oplus A_3) = \begin{bmatrix} 3 & -7 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -2 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 & 7 & 0 & 0 \\ 0 & 0 & 3 & -9 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 7 & -3 \end{bmatrix};$$

d. $\begin{bmatrix} 8 & -2 & -1 & 0 & 0 & 0 \\ 4 & 6 & -7 & 0 & 0 & 0 \\ -3 & 5 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & -2 & -5 \end{bmatrix}$; 6×6

e. Only B, with blocks $B_1 = \begin{bmatrix} 3 & -7 \\ -2 & 4 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 8 & -1 \\ 0 & 5 \end{bmatrix}$.

9. ***Elementary Number Theory:***

a. $A = \begin{bmatrix} 7 & 12 \\ -3 & -5 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} -5 & -12 \\ 3 & 7 \end{bmatrix}$.

b. $A = \begin{bmatrix} a & b \\ y & x \end{bmatrix}$, and $A^{-1} = \begin{bmatrix} x & -b \\ -y & a \end{bmatrix}$ both have only integer entries.

c. $\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}$, with inverse $\begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -7 \\ -7 & 16 \end{bmatrix}$, with inverse $\begin{bmatrix} -16 & -7 \\ -7 & -3 \end{bmatrix}$.

Other answers are possible by switching entries.

Chapter Four Exercises

4.1 Exercises

2. $\dfrac{3}{x+3} + \dfrac{2x+24}{x^2-9} = \dfrac{5}{x-3}$ and $-3 \odot \dfrac{5x-7}{6x+9} = \dfrac{-5x+7}{2x+3}$.
4. a. Yes. b. No. c. Yes. d. No. e. Yes. f. No.
5. Answers:
   a. There are no negatives for the vectors, even though there is a zero vector.
   b. Closed under vector addition, but not closed under scalar multiplication.
   c. There is no zero vector (the zero matrix is not invertible). Also, it is not closed under addition: for example, identity plus its negative yields zero matrix, which is not invertible.
6. Answers:
   a. $-3 \odot \langle 5,-2\rangle = \langle -15,-2\rangle$. All Axioms are valid except for Axiom 7, so this is not a vector space.
   b. $-3 \odot \langle 5,-2\rangle = \langle 15,-6\rangle$. All Axioms are valid except for Axioms 9 and 10, so this is not a vector space.
   c. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 7,4\rangle$. All Axioms are valid except for Axioms 7 and 8, so this is not a vector space.
   d. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 9,-9\rangle$. Invalid axioms: 3, 4, 5, 6, 7 and 8; not a vector space.
   e. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle -9,-3\rangle$. Invalid axioms: 4, 5, 6 and 7; not a vector space.
   f. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 13,-1\rangle$. Invalid axioms: 3, 4, 5, 6, and 7; not a vector space.
   g. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 9,6\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -15,12\rangle$.
      Invalid axioms: 4, 5, 6, 7, 9 and 10; not a vector space.
   h. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 9,6\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -30,6\rangle$.
      Invalid axioms: 4, 5, 6, 7, 9 and 10; not a vector space.
   i. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 3,9\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle 6,-15\rangle$.
      Invalid axioms: 4, 5, 6, 7, 9 and 10; not a vector space.
   j. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle -9,-3\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -15,-6\rangle$.
      Invalid axioms: 4, 5, 6, 7, 9 and 10; not a vector space.
   k. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 9,0\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -15,0\rangle$.
      Invalid axioms: 5, 6, and 10; not a vector space.
   l. $\langle 7,-3\rangle \oplus \langle 2,6\rangle = \langle 7,6\rangle$ and $-3 \odot \langle 5,-2\rangle = \langle -13,3\rangle$.
      Invalid axioms: 8, 9, and 10; not a vector space.
      However, there is a zero vector and negatives:
      $\vec{\mathbf{0}}_V = \langle 2,-3\rangle$ and $\ominus\langle x_1, y_1\rangle = \langle 4-x_1, -6-y_1\rangle$.

4.2 Exercises

1. Answers:

   a. Yes, a member. $-7+19x-47x^2 = 3(6+3x-4x^2)-5(5-2x+7x^2)$

   b. Yes, a member. $105-28x+39x^2+9x^3 = 7(2-4x+5x^3)+13(7+3x^2-2x^3)$

   c. Yes, a member. $\dfrac{2x^2-7x-10}{x^3} = \dfrac{2}{x}-\dfrac{7}{x^2}-\dfrac{10}{x^3}$

   d. Not a member.

   e. Yes, a member. $\dfrac{4x+25}{(x+1)(x-2)} = \dfrac{-7}{x+1}+\dfrac{11}{x-2}$.

   f. Not a member.

3. a. dependent b. independent c. dependent d. independent e. dependent
   f. dependent g. dependent h. dependent i. independent j. independent
   k. independent l. independent m. dependent n. dependent o. dependent
   p. independent q. dependent r. independent s. dependent t. dependent
   u. dependent v. dependent w. dependent x. dependent y. independent.

8. d. independent

4.3 Exercises

9. Answers:

   a. 1/6, −1/6, 7/6, −7/6, 11/6, −11/6, 13/6, 1/7, −1/7, 2/7, −2/7, 3/7, −3/7, 4/7, 1/8, −1/8, 3/8, −3/8, 5/8, −5/8, 7/8

   b. 1/4, −1/3, 3/2, 2, −2, −3/2, 2/3, −1/4, 1/5, 1/6, −1/5, 3/4, −2/3, 5/2, 3, −3, −5/2, 4/3, −3/4, 2/5, −1/6, −1/7, 1/8, −1/7, 5/6.

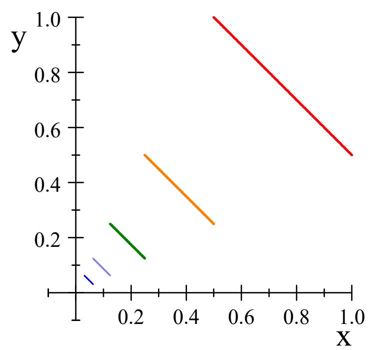
   c. $k = i+j-1$.

10. Answers:

   a. $f(x) = (b-a)x+a$; f. $f(x) = x-a$; h. $f(x) = -x+b$

   k. $f(x) = -x+1$; l. $f(x) = -(x-a)+b = -x+a+b$

   m. $$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ -x+\frac{3}{2} & \text{if } x \in \left(\frac{1}{2},1\right] \\ -x+\frac{3}{4} & \text{if } x \in \left(\frac{1}{4},\frac{1}{2}\right] \\ -x+\frac{3}{8} & \text{if } x \in \left(\frac{1}{8},\frac{1}{4}\right] \\ \vdots & \vdots \\ -x+\frac{3}{2^{n+1}} & \text{if } x \in \left(\frac{1}{2^{n+1}},\frac{1}{2^n}\right] \quad \text{. . . etc.} \end{cases}$$

n. 

Note: the top of each line segment should be an open hole, and the bottom should be a solid dot, and the graph keeps following the pattern as we get closer to the origin, where $f(0) = 0$.

4.4 Exercises

1. a. $E = \{x^{2n} \mid n \in \mathbb{N}\}$ b. *Span*(E) is the set of all even polynomials (those whose graphs are symmetric across the y-axis).
2. a. $O = \{x^{2n+1} \mid n \in \mathbb{N}\}$ b. *Span*(O) is the set of all odd polynomials (those whose graphs are symmetric across the origin).
3. a. $S = \left\{ \frac{1}{x^{n+1}} \mid n \in \mathbb{N} \right\}$

   b. $\frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \cdots + \frac{c_n}{x^n}$

   Note: it makes more sense to start at c_1 instead of c_0.
4. a. $S = (0, \infty)$

   b. $c_1 b_1^x + c_2 b_2^x + \cdots + c_n b_n^x$, where $0 < b_1 < b_2 < \cdots < b_n$.

   d. $f(x) = 1^x = 1$ is a legitimate (constant) function, and we do not care if the functions in S are one-to-one or not.
5. a. $S = \left\{ x^{\frac{1}{n+2}} \mid n \in \mathbb{N} \right\}$

   b. $c_0 x^{1/2} + c_1 x^{1/3} + \cdots + c_n x^{1/(n+2)}$.
6. Answers:
   a. independent
   b. dependent (the logarithm requires a positive base $b \neq 1$).
   c. independent
   d. dependent; $S \subset \mathbb{P}^n$, so once you have $n+2$ of these functions, they are definitely dependent; on the other hand, the set S in (c) is not contained in a single $\mathbb{P}^n$ because there is a polynomial of any degree n in that S.
   e. independent; take a limit at a vertical asymptote to show that the coefficient for that term must be 0.
   f. independent
   g. independent

h. independent
i. independent
j. independent
k. independent
l. independent
m. independent
n. dependent (check out first five vectors)
o. independent

4.5 Exercises

1. Answers:
   a. (iii) $\{6 - x + x^2\}$ (iv) $dim(W) = 1$.
   b. (iii) $\{1 + 2x, -4 + x^2\}$ (iv) $dim(W) = 2$.
   c. (iii) $\{-1 + 2x, -1 + x^2\}$ (iv) $dim(W) = 2$.
   d. (iii) $\{1, -24x - 9x^2 + 5x^3\}$ (iv) $dim(W) = 2$.
   e. (iii) $\{-5 - 7x + 8x^2, 19 - 17x + 2x^3\}$ (iv) $dim(W) = 2$.
   f. (iii) $\{-5 - 7x + 8x^2, 19 - 17x + 2x^3\}$ (iv) $dim(W) = 3$.
   g. (iii) $\{22 - 10x + x^2 + x^3\}$ (iv) $dim(W) = 1$.
2. It does not contain the zero vector, and it is not closed under vector addition, nor scalar multiplication.
3. Answers:
   a. $\{2e^{2x} - 3e^{3x} + e^{5x}\}$; $dim(W_1) = 1$.
   b. $\{-2e^{2x} + e^{3x}, -4e^{2x} + e^{5x}\}$; $dim(W_2) = 2$. W_1 is a subspace of W_2.
   c. Although W_3 contains the zero vector, it is not closed under vector addition, nor scalar multiplication.
4. $\left\{\left(\sqrt{2} - 1\right)\sin(x) + \cos(x), -\sqrt{2}\sin(x) + \tan(x)\right\}$; $dim(W) = 2$.
5. The description states that the roots of $p(x)$ are -1, 1, and 4. Since $p(x)$ is at most cubic, $p(x) = k(x+1)(x-1)(x-4)$, so $dim(W) = 1$ with basis $\{(x+1)(x-1)(x-4)\}$.
8. a. Yes. b. Yes. c. Yes. d. Yes. e. No. (although this is a subset of W, it is dependent) f. Yes. g. No. This polynomial is not in W. h. Yes. i. Yes.
9. $dim(\mathbb{R}^+) = 1$. Did you remember that range of the exponential function b^x is all positive numbers?
14. By the Sum Formula, $\sin(x + k) = \sin(x)\cos(k) + \cos(x)\sin(k)$. Since k is a constant, this is a linear combination of the set $\{\sin(x), \cos(x)\}$, so this (independent) set is a basis for W, and $dim(W) = 2$.
17. d. Possible answer: $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ -7 & 0 \end{bmatrix}\right\}$; it is 2-dimensional.

4.6 Exercises

1. a. lower triangular. b. all of the above. c. symmetric.
   d. all of the above. e. all of the above.

2. a. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ b. symmetric

3. a. symmetric; $\begin{bmatrix} \frac{4}{11} & -\frac{3}{11} \\ -\frac{3}{11} & \frac{5}{11} \end{bmatrix}$ b. upper triangular; $\begin{bmatrix} -\frac{7}{4} & -\frac{35}{18} \\ 0 & \frac{5}{3} \end{bmatrix}$

   c. lower triangular; $\begin{bmatrix} \frac{1}{3} & 0 \\ \frac{7}{24} & \frac{1}{8} \end{bmatrix}$ d. diagonal; $\begin{bmatrix} -\frac{1}{9} & 0 \\ 0 & \frac{4}{3} \end{bmatrix}$

   e. lower triangular; $\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{4}{35} & \frac{1}{7} & 0 \\ \frac{33}{140} & \frac{6}{7} & \frac{3}{4} \end{bmatrix}$ f. symmetric; $\begin{bmatrix} \frac{4}{13} & \frac{3}{13} & \frac{8}{13} \\ \frac{3}{13} & -\frac{1}{13} & \frac{6}{13} \\ \frac{8}{13} & \frac{6}{13} & \frac{29}{13} \end{bmatrix}$

   g. diagonal; $\begin{bmatrix} \frac{5}{3} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & -\frac{9}{7} \end{bmatrix}$ h. upper triangular; $\begin{bmatrix} -\frac{1}{3} & -\frac{1}{2} & -\frac{10}{21} \\ 0 & -\frac{1}{4} & -\frac{2}{7} \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$

   i. lower triangular; $\begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 \\ \frac{2}{45} & \frac{1}{9} & 0 & 0 \\ \frac{5}{3} & \frac{2}{3} & -1 & 0 \\ -\frac{592}{135} & -\frac{44}{27} & \frac{8}{3} & \frac{1}{3} \end{bmatrix}$

   j. symmetric; $\begin{bmatrix} \frac{207}{80} & \frac{1}{4} & \frac{11}{16} & -\frac{139}{80} \\ \frac{1}{4} & 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{11}{16} & \frac{1}{4} & \frac{3}{16} & -\frac{7}{16} \\ -\frac{139}{80} & -\frac{1}{4} & -\frac{7}{16} & \frac{103}{80} \end{bmatrix}$

k. upper triangular; $\begin{bmatrix} \frac{1}{2} & \frac{5}{8} & \frac{51}{56} & -\frac{95}{84} \\ 0 & \frac{1}{4} & \frac{3}{28} & -\frac{1}{7} \\ 0 & 0 & -\frac{1}{7} & \frac{4}{21} \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix}$

l. diagonal; $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & \frac{5}{8} \end{bmatrix}$

4. a. $\begin{bmatrix} -12 & -21 & 9 & -6 & 0 \\ 18 & -4 & 2 & 8 & 12 \\ -35 & -21 & -14 & 63 & 7 \end{bmatrix}$ b. $\begin{bmatrix} -27 & -10 & -14 \\ -3 & 8 & 21 \end{bmatrix}$

e. $T(\langle 5, 8, -6 \rangle) = \langle -15, 16, 42 \rangle$

9. a. $\begin{bmatrix} 24 & 27 & -43 \\ 0 & -6 & 8 \\ 0 & 0 & 28 \end{bmatrix}$

12. a. $T(\vec{e}_1) = 3\vec{e}_1$, $T(\vec{e}_2) = -5\vec{e}_1 + 2\vec{e}_2$, and $T(\vec{e}_3) = 4\vec{e}_1 + \vec{e}_2 - 7\vec{e}_3$.

b. $\vec{v}_1 = \frac{1}{3}\vec{e}_1, \vec{v}_2 = \frac{5}{6}\vec{e}_1 + \frac{1}{2}\vec{e}_2, \vec{v}_3 = \frac{13}{42}\vec{e}_1 + \frac{1}{14}\vec{e}_2 - \frac{1}{7}\vec{e}_3$.

c. $\begin{bmatrix} 1/3 & 5/6 & 13/42 \\ 0 & 1/2 & 1/14 \\ 0 & 0 & -1/7 \end{bmatrix}$

20. a. $AB = \begin{bmatrix} -360 & 342 & -198 \\ 342 & 639 & 342 \\ -198 & 342 & -360 \end{bmatrix} = BA$.

b. $AC = \begin{bmatrix} 178 & -110 & 18 \\ -80 & 1 & -18 \\ -2 & -38 & 216 \end{bmatrix}$; $CA = \begin{bmatrix} 178 & -80 & -2 \\ -110 & 1 & -38 \\ 18 & -18 & 216 \end{bmatrix}$

neither matrix is symmetric.

Chapter Five Exercises

5.1 Exercises

1. a. $21.$ b. $\sqrt{3}$ c. 1
2. a. $\langle 0, 3/5, 1/2\rangle$ b. $\langle 1, 7/25, 1/2\rangle$ c. $\left\langle 0, 3/4, 1/\sqrt{3}\right\rangle$
3. a. $\langle -66, 6\rangle$ b. $\{(x+3)(x-1)\}$ or $\{x^2+2x-3\}$
4. a. $\langle -996, 156, -84\rangle$ b. $\{(x+5)(x-3)(x+2)\}$
5. a. $\langle 117, 13, 18\rangle$ b. $\{z(x)\}$
6. a. $\langle 6, 28, -26\rangle$
7. a. $\langle -33, -2, -10, 16/3\rangle$
8. a. $12x+10$
9. a. $3x^4+2x^3-7x^2$
10. a. x^3+x^2-7x
11. Answers:
    - a. (i) $-5e^{-x}-6e^{2x}$ (iv) $ker(D) = \{z(x)\}$ (v) $range(D) = W.$
    - b. (i) $7e^x\sin(x)+e^x\cos(x)$ (iv) $ker(D) = \{z(x)\}$ (v) $range(D) = W.$
    - c. (i) $3e^{-3x}\sin(2x)+37e^{-3x}\cos(2x)$ (iv) $ker(D) = \{z(x)\}$ (v) $range(D) = W.$
    - d. (i) $33e^{5x}-10xe^{5x}$ (iv) $ker(D) = \{z(x)\}$ (v) $range(D) = W.$
    - e. (i) $20x^2e^{-4x}-18xe^{-4x}+30e^{-4x}$ (iv) $ker(D) = \{z(x)\}$ (v) $range(D) = W.$
    - f. (i) $-4\ln 5x^2 \cdot 5^x+(9(\ln 5)-8)x\cdot 5^x+(9-2(\ln 5))5^x$ (iv) $ker(D) = \{z(x)\}$ (v) $range(D) = W.$
    - g. (i) $6x^2-16x+3$ (iv) $ker(D) = \{1\}$ (v) $\{1, x, x^2\}.$
    - h. (i) $-18x\sin(2x)+8x\cos(2x)-12\sin(2x)-\cos(2x)$ (iv) $ker(D) = \{z(x)\}$ (v) $range(D) = W.$
12. a. $27\sin(x)-\cos(x)$
13. a. $120e^{4x}\sin(3x)+102e^{4x}\cos(3x)$
14. a. $(ac_1-bc_2)e^{ax}\sin bx+(ac_2+bc_1)e^{ax}\cos bx$
15. a. $-4c_1e^{-4x}+3c_2e^{3x}+5c_3e^{5x}$ d. $91c_1e^{-4x}+64c_3e^{5x}$
    e. $\{e^{3x}\}$ f. $\{e^{-4x}, e^{5x}\}$
19. a. $\begin{bmatrix} 4 & 0 \\ -3 & 1 \\ 5 & -7 \end{bmatrix}$

5.2 Exercises

1. a. $\langle -13/2, 19/2, 8\rangle$ c. $\langle -1/2, 1/2, 1\rangle$.
2. b. $\langle 3/2, 27/2, 83, -545/3\rangle$
3. a. $\left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle$ b. $\langle 4/5, 3/5\rangle$ c. $\langle -12/13, 5/13\rangle$ d. $\langle 20/29, 21/29\rangle$
4. a. $\begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & 1 \end{bmatrix}$ c. $\langle 82, 6\rangle$
5. a. $\begin{bmatrix} 1 & -5 & 25 & -125 \\ 1 & 3 & 9 & 27 \\ 1 & -2 & 4 & -8 \end{bmatrix}$ c. $\langle -1285, 179, -91\rangle$
6. a. $\begin{bmatrix} 1 & -5 & 25 \\ 1 & 3 & 9 \\ 1 & -2 & 4 \end{bmatrix}$ c. $\langle 91, 27, 22\rangle$
7. a. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 2 & 1 & 8 \end{bmatrix}$ c. $\langle 6, -33, 59\rangle$
8. a. $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 1 & 1/2 & 1/3 \end{bmatrix}$ c. $\langle 42, 9, 14, 23/6\rangle$
9. a. $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ c. $42x - 16$
10. a. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$ c. $\frac{7}{3}x^2 - \frac{5}{2}x^2 + 4x$
11. Answers:
    a. (i) $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ (ii) $-5e^{-x} - 6e^{2x}$
    b. (i) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (ii) $7e^x \sin(x) + e^x \cos(x)$

c. (i) $\begin{bmatrix} -3 & -2 \\ 2 & -3 \end{bmatrix}$ (ii) $3e^{-3x}\sin(2x) + 37e^{-3x}\cos(2x)$

d. (i) $\begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix}$ (ii) $-10xe^{5x} + 33e^{5x}$

e. (i) $\begin{bmatrix} -4 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & 1 & -4 \end{bmatrix}$ (ii) $20x^2e^{-4x} - 18xe^{-4x} + 30e^{-4x}$

f. (i) $\begin{bmatrix} \ln(5) & 0 & 0 \\ 2 & \ln(5) & 0 \\ 0 & 1 & \ln(5) \end{bmatrix}$

(ii) $-4\ln(5)x^2 \cdot 5^x + (9\ln(5) - 8)x \cdot 5^x + (-2\ln(5) + 9)5^x$

g. (i) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (ii) $6x^2 - 16x + 3$

h. (i) $\begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

(ii) $-18x\sin(2x) + 8x\cos(2x) - 12\sin(2x) - \cos(2x)$

12. b. $\begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}$

13. b. $Diag(k_1, k_2, \ldots, k_n)$ c. a diagonal matrix

14. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

15. b. $\begin{bmatrix} k & 0 & 0 \\ 2 & k & 0 \\ 0 & 1 & k \end{bmatrix}$ c. $kx^n e^{kx} + nx^{n-1}e^{kx}$

16. a. $\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$ b. $27\sin(x) - \cos(x)$ c. $\frac{13}{5}\sin(x) - \frac{9}{5}\cos(x)$

17. a. $\begin{bmatrix} -3 & -15 \\ 15 & -3 \end{bmatrix}$ c. $-96e^{4x}\sin(3x) - 66e^{4x}\cos(3x)$

18. a. $45x^2 + 6x - 20$ d. $\begin{bmatrix} 2 & -1 & 4 & -2 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

19. a. $-11x^3 - 36x^2 + 60x - 41$ d. $\begin{bmatrix} -5 & 3 & 0 \\ 2 & -5 & 6 \\ 0 & 3 & -5 \\ -1 & 2 & 0 \end{bmatrix}$

20. a. $\langle 95, -15, -6 \rangle$. b. $365x - 211$.
c. $T(1) = 4x - 2$, $T(x) = 21x - 7$, and $T(x^2) = 66x - 36$.
d. $[T]_{S,S'} = \begin{bmatrix} -2 & -7 & -36 \\ 4 & 21 & 66 \end{bmatrix}$

21. a. $\langle -11, 3 \rangle$ b. $-46x^2 + 63x + 126$
c. $T(1) = 5x^2 - 6x - 9$; $T(x) = -7x^2 + 11x + 27$
d. $\begin{bmatrix} -9 & 27 \\ -6 & 11 \\ 5 & -7 \end{bmatrix}$

22. a. $\langle 69/2, -14, -3 \rangle$. b. $\frac{311}{2} - 167x + \frac{59}{2}x^2$.
c. $T(1) = \frac{9}{2} - 3x + \frac{1}{2}x^2$, $T(x) = \frac{25}{2} - 10x + \frac{3}{2}x^2$, and $T(x^2) = \frac{83}{2} - 46x + \frac{17}{2}x^2$.
d. $\begin{bmatrix} 9/2 & 25/2 & 83/2 \\ -3 & -10 & -46 \\ 1/2 & 3/2 & 17/2 \end{bmatrix}$

24. $[proj_\Pi] = \frac{1}{122}\begin{bmatrix} 113 & -21 & 24 \\ -21 & 73 & 56 \\ 24 & 56 & 58 \end{bmatrix}$; $[refl_\Pi] = \frac{1}{61}\begin{bmatrix} 52 & -21 & 24 \\ -21 & 12 & 56 \\ 24 & 56 & -3 \end{bmatrix}$;

$[proj_L] = \frac{1}{122}\begin{bmatrix} 9 & 21 & -24 \\ 21 & 49 & -56 \\ -24 & -56 & 64 \end{bmatrix}$

25. Answers:

a. $[proj_\Pi] = \frac{1}{83}\begin{bmatrix} 58 & 15 & -35 \\ 15 & 74 & 21 \\ -35 & 21 & 34 \end{bmatrix}$; $[refl_\Pi] = \frac{1}{83}\begin{bmatrix} 33 & 30 & -70 \\ 30 & 65 & 42 \\ -70 & 42 & -15 \end{bmatrix}$;

$[proj_L] = \frac{1}{83}\begin{bmatrix} 25 & -15 & 35 \\ -15 & 9 & -21 \\ 35 & -21 & 49 \end{bmatrix}$

b. $[proj_\Pi] = \frac{1}{30}\begin{bmatrix} 26 & 2 & -10 \\ 2 & 29 & 5 \\ -10 & 5 & 5 \end{bmatrix}$; $[refl_\Pi] = \frac{1}{15}\begin{bmatrix} 11 & 2 & -10 \\ 2 & 14 & 5 \\ -10 & 5 & -10 \end{bmatrix}$;

$[proj_L] = \frac{1}{30}\begin{bmatrix} 4 & -2 & 10 \\ -2 & 1 & -5 \\ 10 & -5 & 25 \end{bmatrix}$

c. Note: choose $\langle 2,0,3\rangle$ and $\langle 0,1,0\rangle$ as vectors on Π (note that the 2nd vector satisfies the equation);

$[proj_\Pi] = \frac{1}{13}\begin{bmatrix} 4 & 0 & 6 \\ 0 & 13 & 0 \\ 6 & 0 & 9 \end{bmatrix}$; $[refl_\Pi] = \frac{1}{13}\begin{bmatrix} -5 & 0 & 12 \\ 0 & 13 & 0 \\ 12 & 0 & 5 \end{bmatrix}$;

$[proj_L] = \frac{1}{13}\begin{bmatrix} 9 & 0 & -6 \\ 0 & 0 & 0 \\ -6 & 0 & 4 \end{bmatrix}$

26. d. $C = \begin{bmatrix} -c & 0 & a \\ 0 & 1 & 0 \\ a & 0 & c \end{bmatrix}$ is one possible answer.

28. Answers:

a. $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $\vec{w}_3 = 4\vec{w}_1 - 3\vec{w}_2$

b. $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $\vec{w}_3 = -4\vec{w}_1 + 3\vec{w}_2$; $\vec{w}_5 = 2\vec{w}_1 - 5\vec{w}_2 + 7\vec{w}_4$

c. $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_5\}$; $\vec{w}_3 = 4\vec{w}_1 + 9\vec{w}_2$; $\vec{w}_4 = 5\vec{w}_1 + 8\vec{w}_2$; $\vec{w}_6 = -3\vec{w}_1 + 4\vec{w}_2 - 7\vec{w}_5$

d. $S' = \{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$; $\vec{w}_3 = 4\vec{w}_1 - 3\vec{w}_2$; $\vec{w}_5 = 6\vec{w}_1 - 3\vec{w}_2 - 4\vec{w}_4$

30. c. $[S_{\vec{u}}]_{B,B'} = \begin{bmatrix} 0 & -a/c \\ 1 & -b/c \end{bmatrix}$ d. $\begin{bmatrix} 0 & -3/5 \\ 1 & 2/5 \end{bmatrix}$

5.3 Exercises

1. a. No. b. Yes, because $dim(\mathbb{P}^2) < dim(\mathbb{R}^4)$.

   c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_4$

   d. $ker(T) = \{z(x)\}$, so it has no basis, and $nullity(T) = 0$.

   e. $range(T)$ has basis $\{\langle 1,0,0,1\rangle, \langle -2,1,0,1/2\rangle, \langle 4,2,2,1/3\rangle\}$, and $rank(T) = 3$

   f. T is one-to-one but not onto. g. $3 + 0 = 3 = dim(\mathbb{P}^2)$

   h. $p(x) = 4 - 7x + 5x^2$ is the only such polynomial.

2. a. Yes, because $dim(\mathbb{P}^3) > dim(\mathbb{P}^1)$. b. No.

   c. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   d. $ker(T)$ has basis $\{1, x\}$ and $nullity(T) = 2$.

   e. $range(T)$ has basis $\{x^2, x^3\}$ and $rank(T) = 2$.

   f. T is neither one-to-one nor onto. g. $2 + 2 = 4 = dim(\mathbb{P}^3)$

3. a. No. b. Yes, because $dim(\mathbb{P}^2) < dim(\mathbb{P}^3)$.

   c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   d. $ker(T) = \{z(x)\}$, so it has no basis and $nullity(T) = 0$.

   e. $range(T)$ has basis $\{x, x^2, x^3\}$ (we can clear the fractions) and $rank(T) = 3$.

   f. T is one-to-one but not onto. g. $0 + 3 = 3 = dim(\mathbb{P}^2)$.

4. a. Yes, because $dim(\mathbb{P}^3) > dim(\mathbb{P}^2)$. b. No.

   c. $\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   d. $ker(T)$ has basis $\{1 + 2x\}$ and $nullity(T) = 1$.

   e. $range(T)$ has basis $\{2, 4 + 4x, -2 + 6x + 9x^2\}$ or $\{1, x, x^2\}$;
   either basis is acceptable because $rank(T) = 3$.

   f. T is not one-to-one but T is onto. g. $3 + 1 = 4 = dim(\mathbb{P}^3)$

5. a. No. b. Yes, because $dim(\mathbb{P}^2) < dim(\mathbb{P}^3)$.

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d. $ker(T) = \{z(x)\}$, so it has no basis and $nullity(T) = 0$.

e. $range(T)$ has basis $\{-5 + 2x - x^3, 3 - 5x + 3x^2 + 2x^3, 6x - 5x^2\}$ and $rank(T) = 3$.

f. T is one-to-one but not onto. g. $3 + 0 = 3 = \dim(\mathbb{P}^2)$.

6. b. No. c. Yes, because $dim(\mathbb{P}^2) < dim(\mathbb{P}^3)$.

d. $\begin{bmatrix} 0 & -5 & -8 \\ 0 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ e. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

f. $ker(T)$ has basis $\{1\}$ and $nullity(T) = 1$.

g. $range(T)$ has basis $\{-5 + x^2, -8 - 6x + 4x^3\}$ and $rank(T) = 2$.

h. T is neither one-to-one nor onto. i. $2 + 1 = 3 = dim(\mathbb{P}^2)$.

7. b. Yes, because $dim(\mathbb{P}^3) > dim(\mathbb{P}^2)$. c. No.

d. $\begin{bmatrix} 6 & -3 & 6 & -21 \\ -10 & 5 & -10 & 35 \\ 2 & -1 & 2 & -7 \end{bmatrix}$ e. $\begin{bmatrix} 1 & -1/2 & 1 & -7/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

f. $ker(T)$ has basis $\{1 + 2x, -1 + x^2, 7 + 2x^3\}$ and $nullity(T) = 3$.

g. $range(T)$ has basis $\{6 - 10x + 2x^2\}$ and $rank(T) = 1$.

h. T is neither one-to-one nor onto. i. $1 + 3 = 4 = dim(\mathbb{P}^3)$.

8. a. Yes, because $dim(\mathbb{P}^2) > dim(\mathbb{P}^1)$. b. No.

c. $\begin{bmatrix} 1 & 0 & \frac{2}{7} \\ 0 & 1 & -\frac{27}{7} \end{bmatrix}$

d. $ker(T)$ has basis $\{147 - 6x - 7x^2\}$ and $nullity(T) = 1$.

e. $range(T)$ has basis $\{x + 3, 2x - 1\}$ or $\{1, x\}$; either basis is acceptable because $rank(T) = 2$.

f. T is not one-to-one but it is onto. g. $2 + 1 = 3 = dim(\mathbb{P}^2)$.

9. a. No. b. Yes, because $dim(\mathbb{P}^1) < dim(\mathbb{P}^2)$.

c. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

d. $ker(T) = \{z(x)\}$, so it has no basis and $nullity(T) = 0$.

e. $range(T) = Span(\{5x^2 - 6x - 9, 3x^2 - x + 9\})$ and $rank(T) = 2$.

f. T is one-to-one but not onto. g. $2 + 0 = 2 = dim(\mathbb{P}^1)$.

10. a. Yes, because $dim(\mathbb{P}^2) > dim(\mathbb{P}^1)$. b. No.

c. $\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$

d. $ker(T)$ has basis $\{2x + 5, -2x^2 + 2x - 3\}$ and $nullity(T) = 2$.

e. $range(T)$ has basis $\{3x - 7\}$ and $rank(T) = 1$.

f. T is neither one-to-one nor onto. g. $1 + 2 = 3$.

11. a. No. b. Yes, because $dim(\mathbb{P}^1) < dim(\mathbb{P}^2)$.

c. $\begin{bmatrix} 1 & 5/7 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

d. $ker(T) = \{-7x + 2\}$, and $nullity(T) = 1$.

e. $range(T)$ has basis $\{2x^2 + x + 8\}$ and $rank(T) = 1$.

f. T is neither one-to-one nor onto. g. $1 + 1 = 2 = dim(\mathbb{P}^1)$.

12. a. No. b. No.

c. $\begin{bmatrix} 1 & 0 & -\frac{27}{11} \\ 0 & 1 & \frac{14}{11} \\ 0 & 0 & 0 \end{bmatrix}$

d. $ker(T)$ has basis $\{27 - 14x + 11x^2\}$, and $nullity(T) = 1$.

e. $range(T)$ has basis $\{4 - x + 5x^2, 3 + 2x + 12x^2\}$, and $rank(T) = 2$.

h. $p(x) = 3 - 2x + \frac{c_2}{11}(27 - 14x + 11x^2)$ ($\frac{c_2}{11}$ can be replaced by c)

13. a. Yes, because $dim(\mathbb{P}^3) > dim(\mathbb{P}^2)$. b. No.

c. $\begin{bmatrix} 1 & -2 & 0 & 8 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

d. $2(x^3 + 1) + x^2 - 1 = 2x^3 + x^2 + 1$, and
$-8(x^3 + 1) + 5(x + 1) + x - 1 = -8x^3 + 6x - 4$

Don't forget to decode! $nullity(T) = 2$

e. $4(x^2 - 1) - 2(x + 2) + 3(x - 1) = 4x^2 + x - 11$

$7(x^2 - 1) - 5(x + 2) + 6(x - 1) = 7x^2 + x - 23$; $rank(T) = 2$

f. Neither one-to-one nor onto g. $2 + 2 = 4 = dim(\mathbb{P}^3)$.

h. $33x^3 - 18x + 15 + c_2(2x^3 + x^2 + 1) + c_4(-8x^3 + 6x - 4)$

14. b. $[T_1]_{B,B'} = \begin{bmatrix} 4 & -5 & 0 \\ 0 & 7 & -10 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{bmatrix}$, and $[T_2]_{B',B} = \begin{bmatrix} 0 & 3 & -10 & 0 \\ 0 & 0 & 6 & -30 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.

c. The codomain of the first is the same as the domain of the second, in either order.

d. $[T_2 \circ T_1]_{B,B} = \begin{bmatrix} 0 & 11 & -130 \\ 0 & 6 & 0 \\ 0 & 0 & 18 \end{bmatrix}$ and $[T_1 \circ T_2]_{B',B'} = \begin{bmatrix} 0 & 12 & -70 & 150 \\ 0 & 0 & 42 & -300 \\ 0 & 0 & 6 & 60 \\ 0 & 0 & 0 & 18 \end{bmatrix}$

15. b. $[T_1]_{B,B'} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ and $[T_2]_{B',B''} = \begin{bmatrix} 1 & -3 & 9 & -27 \\ 0 & 1 & 4 & 12 \\ 0 & 0 & 2 & -6 \end{bmatrix}$.

c. $[T_2 \circ T_1]_{B,B''} = \begin{bmatrix} -3 & 9 & -27 \\ 2 & 11 & 36 \\ 0 & 4 & -6 \end{bmatrix}$.

d. No, because the codomain of T_2, which is $\mathbb{R}^3$, is not the domain of T_1, which is $\mathbb{P}^2$. The two spaces $\mathbb{R}^3$ and $\mathbb{P}^2$ are both 3-dimensional, but the ***composition*** $T_1 \circ T_2$ is still undefined.

e. Yes, the ***matrix product*** $[T_1]_{B,B'} \cdot [T_2]_{B',B''}$ is a well-defined 4×4 matrix. However, it is completely meaningless in this case.

16. a. No. b. Yes; domain $\mathbb{P}^2$ and codomain $\mathbb{P}^1$. c. $10x^3 - 2x^2 + 16x + 11$

d. $36x - 167$ e. $\begin{bmatrix} 19 & 25 & 33 \\ 2 & -12 & 62 \end{bmatrix}$

17. a. Yes; domain $\mathbb{P}^2$ and codomain $\mathbb{P}^2$. b. Yes; domain $\mathbb{P}^1$ and codomain $\mathbb{P}^1$.

c. $35x^2 - 127x - 11$ d. $41x + 7$

e. $\begin{bmatrix} 11 & -14 \\ 13 & 3 \end{bmatrix}$ g. $220x - 245$ h. $1540x^2 - 5525x - 295$

i. $\begin{bmatrix} -13 & 9 & -16 \\ 32 & -1 & 14 \\ 49 & -7 & 28 \end{bmatrix}$

18. Answers:

a. (i) $[D^2]_B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}$;

(ii) $f''(x) = 5e^{-x} - 12e^{2x}$; $f'''(x) = -5e^{-x} - 24e^{2x}$

b. (i) $[D^2]_B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}$;

(ii) $f''(x) = 5e^{-x} - 12e^{2x}$; $f'''(x) = -2e^{x}\sin(x) + 14e^{x}\cos(x)$

c. (i) $[D^2]_B = \begin{bmatrix} 5 & 12 \\ -12 & 5 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} 9 & -46 \\ 46 & 9 \end{bmatrix}$;

(ii) $f''(x) = -83e^{-3x}\sin(2x) - 105e^{-3x}\cos(2x)$;
$f'''(x) = 459e^{-3x}\sin(2x) + 149e^{-3x}\cos(2x)$

d. (i) $[D^2]_B = \begin{bmatrix} 25 & 0 \\ 10 & 25 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} 125 & 0 \\ 75 & 125 \end{bmatrix}$

(ii) $f''(x) = -50xe^{5x} + 155e^{5x}$; $f'''(x) = -250xe^{5x} + 725e^{5x}$

e. (i) $[D^2]_B = \begin{bmatrix} 16 & 0 & 0 \\ -16 & 16 & 0 \\ 2 & -8 & 16 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} -64 & 0 & 0 \\ 96 & -64 & 0 \\ -24 & 48 & -64 \end{bmatrix}$

(ii) $f''(x) = -80x^2e^{-4x} + 112xe^{-4x} - 138e^{-4x}$;
$f'''(x) = 320x^2e^{-4x} - 608xe^{-4x} + 664e^{-4x}$

f. (i) $[D^2]_B = \begin{bmatrix} (\ln(5))^2 & 0 & 0 \\ 4\ln 5 & (\ln(5))^2 & 0 \\ 2 & 2\ln 5 & (\ln(5))^2 \end{bmatrix}$ and

$$[D^3]_B = \begin{bmatrix} (\ln(5))^3 & 0 & 0 \\ 6(\ln(5))^2 & (\ln(5))^3 & 0 \\ 6\ln 5 & 3(\ln(5))^2 & (\ln(5))^3 \end{bmatrix}$$

(ii) $f''(x) = -4(\ln(5))^2x^2 5^x + \left(9(\ln(5))^2 - 16\ln(5)\right)x5^x + \left(-2(\ln(5))^2 + 18\ln(5) - 8\right)5^x$;

$f'''(x) = -4(\ln(5))^3x^2 5^x + \left(9(\ln(5))^3 - 24(\ln(5))^2\right)x5^x + \left(-2(\ln(5))^3 + 27(\ln(5))^2 - 24\ln(5)\right)5^x$

g. (i) $[D^2]_B = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & -4 & -4 & 0 \\ 4 & 0 & 0 & -4 \end{bmatrix}$ and

$[D^3]_B = \begin{bmatrix} 0 & 8 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ -12 & 0 & 0 & 8 \\ 0 & -12 & -8 & 0 \end{bmatrix}$

(ii) $f''(x) = -16x\sin(2x) - 36x\cos(2x) - 16\sin(2x) - 16\cos(2x)$;

$f'''(x) = 72x\sin(2x) - 32x\cos(2x) + 16\sin(2x) - 68\cos(2x)$

19. a. $[D^2]_B = \begin{bmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{bmatrix}$ and $[D^3]_B = \begin{bmatrix} a^3 - 3ab^2 & b^3 - 3a^2b \\ -b^3 + 3a^2b & a^3 - 3ab^2 \end{bmatrix}$

5.4 Exercises

1. b. $[T]_{B,B'} = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 5 & 25 \\ 0 & 1 & 4 \end{bmatrix}$. c. $[T]^{-1}_{B,B'} = \begin{bmatrix} -\frac{5}{16} & \frac{21}{16} & -\frac{15}{2} \\ -\frac{1}{4} & \frac{1}{4} & -1 \\ \frac{1}{16} & -\frac{1}{16} & \frac{1}{2} \end{bmatrix}$

   d. $p(x) = 9 - 7x + 5x^2$.

2. b. $[T]_{B,B'} = \begin{bmatrix} 1 & -4 & 16 & -64 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & -2 & 3 \end{bmatrix}$. c. $\begin{bmatrix} -\frac{3}{25} & \frac{33}{25} & -\frac{1}{5} & -\frac{6}{5} \\ \frac{19}{175} & -\frac{23}{100} & \frac{17}{140} & \frac{13}{10} \\ \frac{1}{35} & -\frac{1}{10} & \frac{1}{14} & 0 \\ -\frac{3}{175} & \frac{1}{100} & \frac{1}{140} & -\frac{1}{10} \end{bmatrix}$

   d. $p(x) = -11 + 7x - 5x^2 + 2x^3$.

3. a. $5x^2 - 9x + 14$ b. $-3x^2 + 4x + 7$
4. a. $9x^2 - 5x + 17$ b. $-8x^2 - 19x + 23$
5. a. $-4x^2 + 9x - 3$ b. $15x^2 - 8x - 11$
6. a. $-5x^3 + 8x^2 - 3x + 11$ b. $-13x^2 + 7x + 11$
7. a. $-4x^3 + 12x^2 + 19x - 7$ b. $17x^3 - 5x^2 + 12x + 8$
8. a. $-9x^3 + 13x^2 - 5x + 11$ b. $4x^3 - 15x + 8$
9. a. $9x^3 + 7x^2 - 11$ b. $11x^3 - 18x + 9$
10. a. $\frac{2}{3}x^3 - 9x^2 - 11x + 17$ b. $-12x^3 + \frac{7}{4}x^2 + 9x - 3$

11. Answers:

a. (ii) $[D]_B^{-1} = \frac{1}{13}\begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix}$ (iii) $7e^{-3x}\sin(2x) - 5e^{-3x}\cos(2x) + C.$

b. (ii) $[D]_B^{-1} = \frac{1}{25}\begin{bmatrix} 5 & 0 \\ -1 & 5 \end{bmatrix}$ (iii) $3xe^{5x} + 8e^{5x} + C.$

c. (ii) $[D]_B^{-1} = \frac{1}{32}\begin{bmatrix} -8 & 0 & 0 \\ -4 & -8 & 0 \\ -1 & -2 & -8 \end{bmatrix}$ (iii) $4x^2e^{-4x} - 9xe^{-4x} - 3e^{-4x} + C.$

d. (ii) $[D]_B^{-1} = \begin{bmatrix} \frac{1}{\ln 5} & 0 & 0 \\ -\frac{2}{(\ln 5)^2} & \frac{1}{\ln 5} & 0 \\ \frac{2}{(\ln 5)^3} & -\frac{1}{(\ln 5)^2} & \frac{1}{\ln 5} \end{bmatrix}$

(iii)

$$\frac{7}{\ln 5}x^2 \cdot 5^x - \left(\frac{14}{(\ln 5)^2} + \frac{4}{\ln 5}\right)x \cdot 5^x + \left(\frac{14}{(\ln 5)^3} + \frac{4}{(\ln 5)^2} + \frac{9}{\ln 5}\right)5^x + C.$$

e. (ii) $[D]_B^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{bmatrix}$

(iii) $3x\sin(2x) - 7x\cos(2x) - 5\sin(2x) + 6\cos(2x) + C$

f. (ii) $[D]_B^{-1} = \frac{1}{k^2 + m^2}\begin{bmatrix} k & m \\ -m & k \end{bmatrix}$

(iii) $\frac{k}{k^2 + m^2}e^{kx}\sin(mx) - \frac{m}{k^2 + m^2}e^{kx}\cos(mx) + C$ and

$\frac{m}{k^2 + m^2}e^{kx}\sin(mx) + \frac{k}{k^2 + m^2}e^{kx}\cos(mx) + C.$

12. $f(x) = -2x^2e^{-3x} + 8xe^{-3x} + 3e^{-3x}$

13. Answers:

a. (i) $B = \{1, x, x^2\}$ (ii) $T = 3I_3 + 5D - 2D^2$

(iii) $\begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 10 \\ 0 & 0 & 3 \end{bmatrix}$ (iv) $\frac{1}{27}\begin{bmatrix} 9 & -15 & 62 \\ 0 & 9 & -30 \\ 0 & 0 & 9 \end{bmatrix}$

(v) $\frac{1}{3}(2 - 7x + 5x^2)$

b. (i) $B = \{1, x, x^2, x^3\}$ (ii) $T = 3I_4 + 5D - 2D^2$

(iii) $\begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 10 & -12 \\ 0 & 0 & 3 & 15 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ (iv) $\frac{1}{27}\begin{bmatrix} 9 & -15 & 62 & -370 \\ 0 & 9 & -30 & 186 \\ 0 & 0 & 9 & -45 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

(v) $-3511 + 1752x - 747x^2 + 162x^3$

c. (i) $B = \{\sin(x), \cos(x)\}$ (ii) $T = -7I_W + 8D + 3D^2$

(iii) $\begin{bmatrix} -10 & -8 \\ 8 & -10 \end{bmatrix}$ (iv) $\frac{1}{82}\begin{bmatrix} -5 & 4 \\ -4 & -5 \end{bmatrix}$

(v) $-12\sin(x) + 7\cos(x)$

d. (i) $B = \{\sin(x), \cos(x)\}$ (ii) $T = 8I_W + 3D - 4D^2 - 2D^3$

(iii) $\begin{bmatrix} 12 & -5 \\ 5 & 12 \end{bmatrix}$ (iv) $\frac{1}{169}\begin{bmatrix} 12 & 5 \\ -5 & 12 \end{bmatrix}$

(v) $5\sin(x) + 7\cos(x)$

e. (i) $B = \{\sin(2x), \cos(2x)\}$ (ii) $T = -7I_W + 8D + 3D^2$

(iii) $\begin{bmatrix} -19 & -16 \\ 16 & -19 \end{bmatrix}$ (iv) $\frac{1}{617}\begin{bmatrix} -19 & 16 \\ -16 & -19 \end{bmatrix}$

(v) $-5\sin(2x) - 14\cos(2x)$

f. (i) $B = \{\sin(2x), \cos(2x)\}$ (ii) $T = 8I_W + 3D - 4D^2 - 2D^3$

(iii) $\begin{bmatrix} 24 & -22 \\ 22 & 24 \end{bmatrix}$ (iv) $\frac{1}{530}\begin{bmatrix} 12 & 11 \\ -11 & 12 \end{bmatrix}$

(v) $3\sin(2x) - 8\cos(2x)$

g. (i) $B = \{e^{-3x}\sin(2x), e^{-3x}\cos(2x)\}$ (ii) $T = 4I_W + 5D - 9D^2$

(iii) $\begin{bmatrix} -56 & -118 \\ 118 & -56 \end{bmatrix}$ (iv) $\frac{1}{8530}\begin{bmatrix} -28 & 59 \\ -59 & -28 \end{bmatrix}$

(v) $17e^{-3x}\sin(2x) + 11e^{-3x}\cos(2x)$

h. (i) $B = \{e^{-3x}\sin(2x), e^{-3x}\cos(2x)\}$ (ii) $T = -6I_W + 2D + 7D^2 + 3D^3$

(iii) $[T]_B = \begin{bmatrix} 50 & -58 \\ 58 & 50 \end{bmatrix}$ (iv) $\frac{1}{2932}\begin{bmatrix} 25 & 29 \\ -29 & 25 \end{bmatrix}$

(v) $5e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x)$

i. (i) $B = \{xe^{5x}, e^{5x}\}$ (ii) $T = 4I_W - 9D + 2D^2$

(iii) $\begin{bmatrix} 9 & 0 \\ 11 & 9 \end{bmatrix}$ (iv) $\frac{1}{81}\begin{bmatrix} 9 & 0 \\ -11 & 9 \end{bmatrix}$

(v) $4xe^{5x} - 7e^{5x}$

j. (i) $B = \{xe^{5x}, e^{5x}\}$ (ii) $T = 2I_W - 7D - 3D^2 + 4D^3$

(iii) $\begin{bmatrix} 64 & 0 \\ 77 & 64 \end{bmatrix}$ (iv) $\frac{1}{4096}\begin{bmatrix} 64 & 0 \\ -77 & 64 \end{bmatrix}$(v)

(v) $-9xe^{5x} + 13e^{5x}$

k. (i) $B = \{x^2e^{-4x}, xe^{-4x}, e^{-4x}\}$ (ii) $T = 8I_W + 11D + 3D^2$

(iii) $\begin{bmatrix} 12 & 0 & 0 \\ -26 & 12 & 0 \\ 6 & -13 & 12 \end{bmatrix}$ (iv) $\frac{1}{864}\begin{bmatrix} 72 & 0 & 0 \\ 156 & 72 & 0 \\ 133 & 78 & 72 \end{bmatrix}$

(v) $3x^2e^{-4x} + 7xe^{-4x} + 5e^{-4x}$

l. (i) $B = \{x^2e^{-4x}, xe^{-4x}, e^{-4x}\}$ (ii) $T = 11I_W - 8D + 4D^2 + 3D^3$

(iii) $\begin{bmatrix} -85 & 0 & 0 \\ 208 & -85 & 0 \\ -64 & 104 & -85 \end{bmatrix}$ (iv) $\frac{-1}{614125}\begin{bmatrix} 7225 & 0 & 0 \\ 17680 & 7225 & 0 \\ 16192 & 8840 & 7225 \end{bmatrix}$

(v) $2x^2e^{-4x} + 9xe^{-4x} + 7e^{-4x}$

m. (i) $B = \{\sinh(3x), \cosh(3x)\}$ (ii) $T = -8I_W + 9D + 4D^2$

(iii) $\begin{bmatrix} 28 & 27 \\ 27 & 28 \end{bmatrix}$ (iv) $\frac{1}{55}\begin{bmatrix} 28 & -27 \\ -27 & 28 \end{bmatrix}$

(v) $-4\sinh(3x) + 5\cosh(3x)$.

n. (i) $B = \{x\sin(2x),\ x\cos(2x),\ \sin(2x),\ \cos(2x)\}$ (ii) $T = 6I_W + 4D + 3D^2$

(iii) $\begin{bmatrix} -6 & -8 & 0 & 0 \\ 8 & -6 & 0 & 0 \\ 4 & -12 & -6 & -8 \\ 12 & 4 & 8 & -6 \end{bmatrix}$ (iv) $\frac{1}{1250}\begin{bmatrix} -75 & 100 & 0 & 0 \\ -100 & -75 & 0 & 0 \\ 158 & 6 & -75 & 100 \\ -6 & 158 & -100 & -75 \end{bmatrix}$

(v) $-7x\sin(2x) + 5x\cos(2x) + 4\sin(2x) - 6\cos(2x)$

14. d. False.
15. $-8 + 3x - 4x^2 + x^3/2$
16. $-3 + 19x - \frac{3}{2}x^2$
17. $13 - 11x + 8x^2$
18. $9 + 5x - \frac{3}{2}x^2 + 2x^3$
19. $3 + 4x - 7x^3$
20. $9 - 3x - 8x^2 + 5x^3 - 7x^4$
21. a. It is a diagonal matrix where none of the diagonal entries is 0.

    b. $\langle -147, 559/2, -632 \rangle$

c. $[T^{-1}]_{B',B} = Diag(1/3, 2, -1/5)$

d. $-\frac{92}{3} + \frac{74}{5}x + \frac{2}{5}x^2$

22. a. It is a triangular matrix where none of the diagonal entries is 0.

b. $\langle -26, 109/3, -175/3 \rangle$

c. $[T^{-1}]_{B',B} = \begin{bmatrix} -\frac{1}{2} & -\frac{15}{2} & -\frac{37}{2} \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix}$ d. $79 + 44x + 2x^2$

23. a. $\frac{1}{20}\begin{bmatrix} -5 & 5 & 5 \\ 8 & 4 & -4 \\ 1 & 3 & 7 \end{bmatrix}$ b. $34 - 29x + 7x^2$.

30. c. $dim(Hom(V, W)) = m \cdot n$.

33. a. Both are 1-dimensional.

b. Both are 1-dimensional.

c. Both are 2-dimensional

d. Both are n-dimensional.

e. Both have dimension $n(n+1)/2$.
The transpose operation is an isomorphism.

f. $\dim(Sym(n)) = n(n+1)/2$ as well.

g. Both have dimension k^2.

Chapter Six Exercises

6.1 Exercises

1. a. $\{\langle 1,-1,-12,6\rangle,\langle 11,-16,13,1\rangle,\langle 1,1,-16,10\rangle\}$; $dim(V\vee W)=3$;
b. $\{\langle 5,-7,-2,4\rangle\}$; $dim(V\cap W)=1$,
c. $\langle 5,-7,-2,4\rangle=\frac{3}{5}\langle 1,-1,-12,6\rangle+\frac{2}{5}\langle 11,-16,13,1\rangle$, and
$\langle 5,-7,-2,4\rangle=\frac{1}{3}\langle 1,1,-16,10\rangle+\frac{2}{3}\langle 7,-11,5,1\rangle$. d. $3=2+2-1$.
2. a. $\{\langle 3,5,-2,4\rangle,\langle 1,2,7,-3\rangle,\langle 0,2,1,-5\rangle,\langle 2,-3,1,6\rangle\}$; $dim(V\vee W)=4$ i.e. $V\vee W=\mathbb{R}^4$.
b. $dim(V\cap W)=0$, so it has no basis. d. $4=2+2-0$
3. a. $\{\langle -3,-2,7,-4\rangle,\langle -2,13,-12,-2\rangle,\langle -2,3,-5,1\rangle,\langle -3,-5,6,-11\rangle\}$;
$dim(V\vee W)=4$ i.e. $V\vee W=\mathbb{R}^4$.
b. $\{\langle -26,-17,14,0\rangle,\langle 3,-8,0,7\rangle\}$; $dim(V\cap W)=2$
c. $\langle -26,-17,14,0\rangle=4\langle -3,-2,7,-4\rangle-3\langle -2,13,-12,-2\rangle+10\langle -2,3,-5,1\rangle$, and
$\langle 3,-8,0,7\rangle=-\langle -3,-2,7,-4\rangle-\langle -2,13,-12,-2\rangle+\langle -2,3,-5,1\rangle$;
$\langle -26,-17,14,0\rangle=4\langle -3,-5,6,-11\rangle-3\langle -1,16,-8,8\rangle-17\langle 1,-3,2,-4\rangle$, and
$\langle 3,-8,0,7\rangle=-\langle -3,-5,6,-11\rangle-\langle -1,16,-8,8\rangle-\langle 1,-3,2,-4\rangle$. d. $4=3+3-2$.
4. a. $\{\langle -3,4,-1,4,6\rangle,\langle -6,8,5,15,-13\rangle,\langle 1,-2,0,-5,3\rangle,\langle 1,3,-2,7,2\rangle\}$; $dim(V\vee W)=4$.
b. $\{\langle 3,-2,-5,0,4\rangle\}$; $dim(V\cap W)=1$.
c. $\langle 3,-2,-5,0,4\rangle=0\langle -3,4,-1,4,6\rangle-\langle -6,8,5,15,-13\rangle-3\langle 1,-2,0,-5,3\rangle$, and
$\langle 3,-2,-5,0,4\rangle=-\langle 1,3,-2,7,2\rangle-\langle -4,-1,7,-7,-6\rangle$. d. $4=3+2-1$.
5. a. $\{\langle -1,7,5,-6,6\rangle,\langle -1,-8,2,-4,2\rangle,\langle 1,0,3,-4,3\rangle,\langle 5,3,-2,7,-4\rangle,\langle -6,9,-2,0,0\rangle\}$;
$dim(V\vee W)=5$ i.e. $V\vee W=\mathbb{R}^5$.
b. $\{\langle -17,31,-3,0,4\rangle,\langle -3,7,-1,2,0\rangle\}$; $dim(V\cap W)=2$.
c. $\langle -17,31,-3,0,4\rangle=3\langle -1,7,5,-6,6\rangle-2\langle -1,-8,2,-4,2\rangle$
$-6\langle 1,0,3,-4,3\rangle-2\langle 5,3,-2,7,-4\rangle$, and
$\langle -3,7,-1,2,0\rangle=\langle -1,7,5,-6,6\rangle-2\langle 1,0,3,-4,3\rangle$;
$\langle -17,31,-3,0,4\rangle=3\langle -6,9,-2,0,0\rangle-2\langle -5,1,-3,-3,-2\rangle+3\langle -3,2,-1,-2,0\rangle$, and
$\langle -3,7,-1,2,0\rangle=\langle -6,9,-2,0,0\rangle-\langle -3,2,-1,-2,0\rangle$. d. $5=4+3-2$.
6. a. $\{6-x+2x^2+10x^3,11-3x+6x^2+2x^3,3+17x+5x^2+4x^3\}$; $dim(V\vee W)=3$
b. $\{-3+x-2x^2+2x^3\}$; $dim(V\cap W)=1$.
c. $-3+x-2x^2+2x^3=\frac{2}{7}(6-x+2x^2+10x^3)-\frac{3}{7}(11-3x+6x^2+2x^3)$, and
$-3+x-2x^2+2x^3=2(3+17x+5x^2+4x^3)-3(3+11x+4x^2+2x^3)$
d. $3=2+2-1$.
7. a. $\{2+5x-10x^2+5x^3,6-7x-4x^2+x^3,-8+14x-16x^2+3x^3\}$; $dim(V\vee W)=4$,
i.e. $V\vee W=\mathbb{P}^3$. b. $\{4-x-7x^2+3x^3\}$; $dim(V\cap W)=1$.
c. $4-x-7x^2+3x^3=\frac{1}{2}(2+5x-10x^2+5x^3)+\frac{1}{2}(6-7x-4x^2+x^3)$, and
$4-x-7x^2+3x^3=\frac{3}{5}(2+3x-19x^2+13x^3)+\frac{2}{5}(7-7x+11x^2-12x^3)$
d. $4=3+2-1$.
8. a. $\{-3-2x+4x^2+x^4,6-3x^2+5x^3-5x^4,-7-7x+8x^2+2x^3+8x^4,$
$-5-2x+7x^2+x^3-x^4,1-6x+3x^2-2x^3-4x^4\}$; $dim(V\vee W)=5$, i.e. $V\vee W=\mathbb{P}^4$.
b. $\{5008+9057x-12636x^2,28-33x+52x^3,18+15x-52x^4\}$; $dim(V\cap W)=3$.
c. $56-x-52x^2=30(-3-2x+4x^2+x^4)+5(6-3x^2+5x^3-5x^4)-$
$3(-7-7x+8x^2+2x^3+8x^4)-19(-5-2x+7x^2+x^3-x^4)$;

$$28-33x+52x^3 = 2(-3-2x+4x^2+x^4)+9(6-3x^2+5x^3-5x^4)+ 5(-7-7x+8x^2+2x^3+8x^4)-3(-5-2x+7x^2+x^3-x^4), \text{ and}$$
$$18+15x-52x^4 = 18(-3-2x+4x^2+x^4)+3(6-3x^2+5x^3-5x^4)- 7(-7-7x+8x^2+2x^3+8x^4)-(-5-2x+7x^2+x^3-x^4);$$
$$56-x-52x^2 = -26(1-6x+3x^2-2x^3-4x^4)+5(5-14x+7x^2+x^3-12x^4)- 3(-9x+3x^2+2x^4)-19(-3+6x+3x^3+2x^4);$$
$$28-33x+52x^3 = -26(1-6x+3x^2-2x^3-4x^4)+9(5-14x+7x^2+x^3-12x^4)+ 5(-9x+3x^2+2x^4)-3(-3+6x+3x^3+2x^4);$$
$$18+15x-52x^4 = 3(5-14x+7x^2+x^3-12x^4)-7(-9x+3x^2+2x^4)- (-3+6x+3x^3+2x^4)$$

d. $5 = 4+4-3$.

11. $6 \le dim(V \cap W) \le 8$. 13. W must be a subspace of V.

10. a. W must be a subspace of V.

c. $V \cap W = W$, or $V \vee W = V$.

6.2 Exercises

1. a. $\{\langle 1,0,4\rangle,\langle 0,1,-7\rangle\}$ b. $\{\langle 3,5,4,-1\rangle,\langle 2,3,2,-1\rangle\}$ c. $\begin{bmatrix} 17 & -28 \\ -28 & 50 \end{bmatrix}$ d. $\begin{bmatrix} \frac{25}{33} & \frac{14}{33} \\ \frac{14}{33} & \frac{17}{66} \end{bmatrix}$

2. a. $\{\langle 1,-3,0\rangle,\langle 0,0,1\rangle\}$ b. $\{\langle 2,-3,-4,5\rangle,\langle -7,-1,9,3\rangle\}$ c. $\begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix}$

3. a. $\{\langle 1,0,0\rangle,\langle 0,1,0\rangle,\langle 0,0,1\rangle\}$ b. $\{\langle 2,-3,-4,5\rangle,\langle -6,9,12,-5,\rangle,\langle -7,-1,9,3\rangle\}$ c. I_3, with inverse d. I_3. Note, though that this is not the identity transformation.

4. a. $\{\langle 1,0,-2,-2\rangle,\langle 0,1,2,1\rangle\}$ b. $\{\langle 3,2,-2\rangle,\langle 5,3,-1\rangle\}$ c. $\begin{bmatrix} 9 & -6 \\ -6 & 6 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}$

5. a. $\{\langle 1,0,-2\rangle,\langle 0,1,1\rangle\}$ b. $\{\langle 3,2,-2\rangle,\langle 5,3,-1\rangle\}$ c. $\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{6} \end{bmatrix}$

6. a. $\{\langle 1,5,0,4\rangle,\langle 0,0,1,-3\rangle\}$ b. $\{\langle 2,3,-4\rangle,\langle 5,7,-9\rangle\}$ c. $\begin{bmatrix} 42 & -12 \\ -12 & 10 \end{bmatrix}$ d. $\begin{bmatrix} \frac{5}{138} & \frac{1}{23} \\ \frac{1}{23} & \frac{7}{46} \end{bmatrix}$

7. a. $\{\langle 1,5,0,0\rangle,\langle 0,0,1,0\rangle,\langle 0,0,0,1\rangle\}$ b. $\{\langle 2,3,-4\rangle,\langle 5,7,-9\rangle,\langle -7,-9,8\rangle\}$

c. $\begin{bmatrix} 26 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{26} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. a. $\{\langle 1,4,0,-5,2\rangle,\langle 0,0,1,6,-3\rangle\}$ b. $\{\langle -5,3,2,-4\rangle,\langle -4,-2,3,1\rangle\}$

c. $\begin{bmatrix} 46 & -36 \\ -36 & 46 \end{bmatrix}$ d. $\begin{bmatrix} \frac{23}{410} & \frac{9}{205} \\ \frac{9}{205} & \frac{23}{410} \end{bmatrix}$

9. a. $\{\langle 1,4,0,-5\rangle,\langle 0,0,1,6\rangle\}$ b. $\{\langle -5,3,2,-4\rangle,\langle -4,-2,3,1\rangle\}$
c. $\begin{bmatrix} 42 & -30 \\ -30 & 37 \end{bmatrix}$ d. $\begin{bmatrix} \frac{37}{654} & \frac{5}{109} \\ \frac{5}{109} & \frac{7}{109} \end{bmatrix}$

10. a. $\{\langle 1,-2,6,0,-4\rangle,\langle 0,0,0,1,5\rangle\}$ b. $\{\langle -5,2,-3,4\rangle,\langle -3,-3,2,-5\rangle\}$
c. $\begin{bmatrix} 57 & -20 \\ -20 & 26 \end{bmatrix}$ d. $\begin{bmatrix} \frac{13}{541} & \frac{10}{541} \\ \frac{10}{541} & \frac{57}{1082} \end{bmatrix}$

11. a. $\{\langle 1,-3,0,0,2\rangle,\langle 0,0,1,0,-4\rangle,\langle 0,0,0,1,7\rangle\}$ b. $\{\langle -2,3,4,-5\rangle,\langle -3,7,-1,2\rangle,\langle -5,4,-2,3\rangle\}$
c. $\begin{bmatrix} 14 & -8 & 14 \\ -8 & 17 & -28 \\ 14 & -28 & 50 \end{bmatrix}$ d. $\begin{bmatrix} \frac{33}{332} & \frac{1}{83} & -\frac{7}{332} \\ \frac{1}{83} & \frac{63}{83} & \frac{35}{83} \\ -\frac{7}{332} & \frac{35}{83} & \frac{87}{332} \end{bmatrix}$

12. a. $\{\langle 1,-3,0,0,0\rangle,\langle 0,0,1,0,0\rangle,\langle 0,0,0,1,0\rangle,\langle 0,0,0,0,1\rangle\}$
b. $\{\langle -2,3,4,-5\rangle,\langle -3,7,-1,2\rangle,\langle -5,4,-2,3\rangle,\langle -7,6,-2,3\rangle\}$
c. $\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. a. $\{\langle 1,5,0,-2\rangle,\langle 0,0,1,6\rangle\}$ b. $\{\langle 2,-4,3,5,-6\rangle,\langle -1,1,1,-3,2\rangle\}$
c. $\begin{bmatrix} 30 & -12 \\ -12 & 37 \end{bmatrix}$ d. $\begin{bmatrix} \frac{37}{966} & \frac{2}{161} \\ \frac{2}{161} & \frac{5}{161} \end{bmatrix}$

14. a. $\{\langle 1,5,0,0\rangle,\langle 0,0,1,0\rangle,\langle 0,0,0,1\rangle\}$
b. $\{\langle 2,-4,3,5,-6\rangle,\langle -1,1,1,-3,2\rangle,\langle -9,14,0,-28,24\rangle\}$
c. $\begin{bmatrix} 26 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} \frac{1}{26} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

15. a. $\{\langle 1,0,0,-2\rangle,\langle 0,1,0,3\rangle,\langle 0,0,1,-5\rangle\}$
b. $\{\langle -2,5,1,-2,-1\rangle,\langle 1,-1,1,-2,1\rangle,\langle 1,-1,-1,1,1\rangle\}$
c. $\begin{bmatrix} 5 & -6 & 10 \\ -6 & 10 & -15 \\ 10 & -15 & 26 \end{bmatrix}$ d. $\begin{bmatrix} \frac{35}{39} & \frac{2}{13} & -\frac{10}{39} \\ \frac{2}{13} & \frac{10}{13} & \frac{5}{13} \\ -\frac{10}{39} & \frac{5}{13} & \frac{14}{39} \end{bmatrix}$

16. a. The standard basis for $\mathbb{R}^4$ b. The four columns of $[T]$. c. I_3 d. I_3. Again, this is ***not*** the identity transformation.

6.3 Exercises

1. a. $\{\langle -5,1,12,11\rangle\}$ b. $\{\langle 1,1,0\rangle,\langle -4,7,1\rangle\}$
2. a. $\{\langle -7,-1,9,3\rangle,\langle 2,-3,-4,5\rangle\}$ (scaled down) b. $\{\langle 2,0,1\rangle,\langle 3,1,0\rangle,\langle 0,0,1\rangle\}$
3. a. $\{\langle -7,-1,9,7\rangle,\langle 2,-3,-4,3\rangle\}$ (scaled down) b. $\{\langle 1,0,1\rangle,\langle 3,1,0\rangle\}$ (scaled up)
4. a. $\{\langle 3,2,-2\rangle,\langle 5,2,6\rangle\}$ (scaled down) b. $\{\langle -1,1,0,0\rangle,\langle 2,1,0,0\rangle,\langle 2,-2,1,0\rangle,\langle 2,-1,0,1\rangle\}$
5. a. $\{\langle 5,4,-8\rangle,\langle 3,2,-2\rangle\}$ b. $\{\langle -1,1,0\rangle,\langle 2,-1,1\rangle,\langle 1,-1,1\rangle\}$
6. a. $\{\langle 4,7,-10\rangle,\langle 13,17,-21\rangle\}$ b. $\{\langle -1,0,1,0\rangle,\langle -5,1,0,0\rangle,\langle -4,0,3,1\rangle\}$
7. a. $\{\langle 8,10,-9\rangle,\langle -9,-14,16\rangle,\langle 23,38,-65\rangle\}$ or simply $\{\vec{i},\vec{j},\vec{k}\}$
   b. $\{\langle -1,0,1,0\rangle,\langle -15,0,5,-1\rangle,\langle -5,1,0,0\rangle\}$
8. a. $\{\langle -43,105,-8,-110\rangle,\langle -9,1,5,-3\rangle\}$ (scaled down)
   b. $\{\langle -1,0,1,0,0\rangle,\langle -4,1,0,0\rangle,\langle 5,0,-6,1,0\rangle,\langle -2,0,3,0,1\rangle\}$
9. a. $\{\langle -88,22,45,-41\rangle,\langle -92,350,-57,-355\rangle\}$
   b. $\{\langle 1,0,1,0\rangle,\langle -4,1,0,0\rangle,\langle 5,0,-6,1\rangle\}$
10. a. $\{\langle 192,3,43,-13\rangle\}$ (scaled down)
    b. $\{\langle 1,0,0,-1,0\rangle,\langle 2,1,0,0,0\rangle,\langle -6,0,1,0,0\rangle,\langle 4,0,0,-5,1\rangle\}$
11. a. $\{\langle 15,13,45,-58\rangle\}$ b. $\{\langle 3,1,0,0,0\rangle,\langle -2,0,4,-7,1\rangle\}$
12. a. $\{\langle -88,97,7,-1\rangle,\langle 4,-40,-79,100\rangle\}$
    b. $\left\{\left\langle -\frac{1}{10},0,\frac{1}{5},-\frac{7}{20},\frac{1}{20}\right\rangle,\left\langle -\frac{1}{10},0\frac{1}{5},-\frac{17}{20},\frac{11}{20}\right\rangle,\left\langle \frac{9}{10},0,\frac{1}{5},\frac{143}{20},-\frac{109}{20}\right\rangle,\langle 3,1,0,0,0\rangle\right\}$
13. a. $\{\langle 12,-22,13,31,-34\rangle,\langle -61,85,1,-171,146\rangle\}$ b. $\{\langle 2,0,1,0\rangle,\langle -5,1,0,0\rangle,\langle 2,0,-6,1\rangle\}$
14. a. $\{\langle -21,31,3,-65,54\rangle,\langle -53,78,8,-164,136\rangle\}$
    b. $\left\{\langle 2,0,-6,1\rangle,\langle -\frac{3}{4},0,\frac{3}{4},-\frac{1}{4}\rangle,\left\langle -\frac{1}{8},0,-\frac{11}{8},-\frac{1}{8}\right\rangle,\langle -5,1,0,0\rangle\right\}$.
    Since this basis has 4 elements, the preimage is all of $\mathbb{R}^4$, so any basis for $\mathbb{R}^4$ is also a correct answer, including the standard basis.
15. a. $\{\langle -6,21,-11,12,-1\rangle,\langle 7,-25,19,-23,1\rangle,\langle -2,14,-12,13,2\rangle\}$
    b. $\{\langle 0,2,-1,0,0\rangle,\langle 2,-3,5,1\rangle\}$

6.4 Exercises

1. Yes. 2. No. 3. Yes. 4. No. 5. Yes. 6. No. 7. Yes. 8. Yes. 9. No. 10. Yes.
11. $x_0 = -21$ and $z_0 = 14$. 12. $x_0 = 30$, $y_0 = -45$, and $z_0 = 18$.
13. a. $\{\langle 3,-1,2,0\rangle,\vec{e}_1,\vec{e}_2,\vec{e}_4\}$ b. $\{\vec{e}_1,\vec{e}_2,\vec{e}_4\}$ c. 3; d. $3 = 4-1$
14. a. $\{\langle 3,5,2,-2\rangle,\langle -2,1,2,-2\rangle,\vec{e}_1,\vec{e}_3\}$ b. $\{\vec{e}_1,\vec{e}_3\}$ c. 2; d. $2 = 4-2$
15. a. $\{\langle 3,0,-2,0,7\rangle,\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4\}$ b. $\{\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4\}$ c. 4; d. $4 = 5-1$
16. a. $\{\langle 2,0,7,3,0\rangle,\langle 0,5,-14,-6,0\rangle,\vec{e}_1,\vec{e}_3,\vec{e}_5\}$ b. $\{\vec{e}_1,\vec{e}_3,\vec{e}_5\}$ c. 3; d. $3 = 5-2$
17. a. $\{\langle 4,-3,0,0,5\rangle,\langle 2,-3,0,0,5\rangle,\langle 2,1,0,0,5\rangle,\vec{e}_3,\vec{e}_4\}$ b. $\{\vec{e}_3,\vec{e}_4\}$ c. 2 d. $2 = 5-3$

6.5 Exercises

1. a. $\{\langle 3,5,4,-1\rangle,\langle 2,3,2,-1\rangle\}$ b. $\{\langle -4,7,1\rangle\}$ c. $\{\vec{e}_1 + ker(T),\vec{e}_2 + ker(T)\}$
   d. $\widetilde{T}(\vec{e}_1 + ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_2 + ker(T)) = \vec{c}_2$
2. a. $\{\langle 2,-3,-4,5\rangle,\langle -7,-1,9,3\rangle\}$ b. $\{\langle 3,1,0\rangle\}$ c. $\{\vec{e}_1 + ker(T),\vec{e}_3 + ker(T)\}$
   d. $\widetilde{T}(\vec{e}_1 + ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_3 + ker(T)) = \vec{c}_3$
3. a. $\{\langle 3,2,-2\rangle,\langle 5,3,-1\rangle\}$ b. $\{\langle 2,-2,1,0\rangle,\langle 2,-1,0,1\rangle\}$ c. $\{\vec{e}_1 + ker(T),\vec{e}_2 + ker(T)\}$
   d. $\widetilde{T}(\vec{e}_1 + ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_2 + ker(T)) = \vec{c}_2$
4. a. $\{\langle 3,2,-2\rangle,\langle 5,3,-1\rangle\}$ b. $\{\langle 2,1,0\rangle\}$ c. $\{\vec{e}_1 + ker(T),\vec{e}_2 + ker(T)\}$
   d. $\widetilde{T}(\vec{e}_1 + ker(T)) = \vec{c}_1; \widetilde{T}(\vec{e}_2 + ker(T)) = \vec{c}_2$

5. a. $\{\langle 2,3,-4\rangle,\langle 5,7,-9\rangle\}$ b. $\{\langle -5,1,0,0\rangle,\langle -4,3,0,1\rangle\}$ c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3$
6. a. $\{\langle 2,3,-4\rangle,\langle 5,7,-9\rangle,\langle -7,-9,8\rangle\}$ b. $\{\langle -5,1,0,0\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T),\vec{e}_4+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3;\widetilde{T}(\vec{e}_4+ker(T))=\vec{c}_4$
7. a. $\{\langle -5,3,2,-4\rangle,\langle -4,-2,3,1\rangle\}$ b. $\{\langle -4,1,0,0,0\rangle,\langle 5,0,-6,1,0\rangle,\langle -2,0,3,0,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3$
8. a. $\{\langle -5,3,2,-4\rangle,\langle -4,-2,3,1\rangle\}$ b. $\{\langle -4,1,0,0,0\rangle,\langle 5,0,-6,1,0\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3$
9. a. $\{\langle -5,2,-3,4\rangle,\langle -3,-3,2,5\rangle\}$ b. $\{\langle 2,1,0,0,0\rangle,\langle -6,0,1,0,0\rangle,\langle 4,0,0,-5,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_4+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_4+ker(T))=\vec{c}_4$
10. a. $\{\langle -2,3,4,-5\rangle,\langle -3,7,-1,2\rangle,\langle -5,4,-2,3\rangle\}$ b. $\{\langle 3,1,0,0,0\rangle,\langle -2,0,4,-7,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T),\vec{e}_4+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3$
$\widetilde{T}(\vec{e}_4+ker(T))=\vec{c}_4$
11. a. $\{\langle -2,3,4,-5\rangle,\langle -3,7,-1,2\rangle,\langle -5,4,-2,3\rangle,\langle -7,6,-2,3\rangle\}$ b. $\{\langle 3,1,0,0,0\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T),\vec{e}_4+ker(T),\vec{e}_5+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3;\widetilde{T}(\vec{e}_4+ker(T))=\vec{c}_4;\widetilde{T}(\vec{e}_5+ker(T))=\vec{c}_5$
12. a. $\{\langle 2,-4,3,5,-6\rangle,\langle -1,1,1,-3,2\rangle\}$ b. $\{\langle -5,1,0,0\rangle,\langle 2,0,-6,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3$
13. a. $\{\langle 2,-4,3,5,-6\rangle,\langle -1,1,1,-3,2\rangle,\langle -9,14,0,-28,24\rangle\}$ b. $\{\langle -5,1,0,0\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_3+ker(T),\vec{e}_4+ker(T)\}$
d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;\widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3;\widetilde{T}(\vec{e}_4+ker(T))=\vec{c}_4$
14. a. $\{\langle -2,5,1,-2,-1\rangle,\langle 1,-1,1,-2,1\rangle,\langle 1,-1,-1,1,1\rangle\}$ b. $\{\langle 2,-3,5,1\rangle\}$
c. $\{\vec{e}_1+ker(T),\vec{e}_2+ker(T),\vec{e}_3+ker(T)\}$ d. $\widetilde{T}(\vec{e}_1+ker(T))=\vec{c}_1;$
$\widetilde{T}(\vec{e}_2+ker(T))=\vec{c}_2;\ \widetilde{T}(\vec{e}_3+ker(T))=\vec{c}_3$
15. a. $\{\langle\langle -2,-1,1\rangle\rangle+U\}$ b. $\{\vec{e}_2+W\}$ c. $\{\langle -2,-1,1\rangle+U,\vec{e}_2+U\}$ d. $\{\vec{e}_2+W/U\}$
e. $\widetilde{T}(\vec{e}_2+U+W/U)=\vec{e}_2+W$
16. a. $\{\langle 1,1,1,2\rangle+U\}$ b. $\{\vec{e}_1+W,\vec{e}_3+W\}$ c. $\{\langle 1,1,1,2\rangle+U,\vec{e}_1+U,\vec{e}_3+U\}$
d. $\{\vec{e}_1+W/U,\vec{e}_3+W/U\}$ e. $\widetilde{T}(\vec{e}_1+U+W/U)=\vec{e}_1+W;\ \widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W;$
17. a. $\{\langle 1,1,1,2\rangle+U,\langle 3,-1,1,2\rangle+U\}$ b. $\{\vec{e}_3+W\}$
c. $\{\langle 1,1,1,2\rangle+U,\langle 3,-1,1,2\rangle+U,\vec{e}_3+U\}$
d. $\{\vec{e}_3+W/U\}$ e. $\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W$
18. a. $\{\langle 3,-1,1,2\rangle+U\}$ b. $\{\vec{e}_3+W\}$ c. $\{\langle 3,-1,1,2\rangle+U,\vec{e}_3+U\}$ d. $\{\vec{e}_3+W/U\}$
e. $\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W$
19. a. $\{\langle 1,1,1,2,-3\rangle+U,\langle 3,-1,1,2,-3\rangle+U\}$ b. $\{\vec{e}_3+W,\vec{e}_4+W\}$
c. $\{\langle 1,1,1,2,-3\rangle+U,\langle 3,-1,1,2,-3\rangle+U,\vec{e}_3+U,\vec{e}_4+U\}$
d. $\{\vec{e}_3+W/U,\vec{e}_4+W/U\}$ e. $\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W;\widetilde{T}(\vec{e}_4+U+W/U)=\vec{e}_4+W$
20. a. $\{\langle 3,-1,1,2,-3\rangle+U\}$ b. $\{\vec{e}_3+W,\vec{e}_4+W\}$ c. $\{\langle 3,-1,1,2,-3\rangle+U,\vec{e}_3+U,\vec{e}_4+U\}$
d. $\{\vec{e}_3+W/U,\vec{e}_4+W/U\}$ e. $\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W;\widetilde{T}(\vec{e}_4+U+W/U)=\vec{e}_4+W$
21. a. $\{\langle 3,-1,1,2,-3\rangle+U,\langle 3,-1,1,-1,-3\rangle+U\}$ b. $\{\vec{e}_3+W\}$
c. $\{\langle 3,-1,1,2,-3\rangle+U,\langle 3,-1,1,-1,-3\rangle+U,\vec{e}_3+U\}$ d. $\{\vec{e}_3+W/U\}$
e. $\widetilde{T}(\vec{e}_3+U+W/U)=\vec{e}_3+W$
22. a. $\{\langle 1,-1,-12,6\rangle,\langle 11,-16,13,1\rangle,\langle 1,1,-16,10\rangle\}$ b. $\{\langle 5,-7,-2,4\rangle\}$
c. $\{\langle 1,-1,-12,6\rangle+W,\langle 1,1,-16,10\rangle+W\}$
d. $\{\langle 1,-1,-12,6\rangle+(V\cap W),\langle 1,1,-16,10\rangle+(V\cap W)\}$

e. $\{\langle 1,-1,-12,6\rangle + V, \langle 1,1,-16,10\rangle + V\}$
f. $\{\langle 1,-1,-12,6\rangle + (V \cap W), \langle 1,1,-16,10\rangle + (V \cap W)\}$
g. $\widetilde{T}_1(\langle 1,-1,-12,6\rangle + W) = \langle 1,-1,-12,6\rangle + (V \cap W);$
$\widetilde{T}_1(\langle 1,1,-16,10\rangle + W) = \langle 1,1,-16,10\rangle + (V \cap W);$
h. $\widetilde{T}_2(\langle 1,-1,-12,6\rangle + V) = \langle 1,-1,-12,6\rangle + (V \cap W)$
$\widetilde{T}_2(\langle 1,1,-16,10\rangle + V) = \langle 1,1,-16,10\rangle + (V \cap W)$

23. a. $\{\langle 3,5,-2,4\rangle, \langle 1,2,7,-3\rangle, \langle 0,2,1,-5\rangle, \langle 2,-3,1,6\rangle\}$
b. $dim(V \cap W) = 0$, so it has no basis.
c. $\{\langle 3,5,-2,4\rangle + W, \langle 1,2,7,-3\rangle + W\}$
d. $\left\{\langle 3,5,-2,4\rangle + \left\{\vec{0}_4\right\}, \langle 1,2,7,-3\rangle + \left\{\vec{0}_4\right\}\right\}$
e. $\{\langle 0,2,1,-5\rangle + V, \langle 2,-3,1,6\rangle + V\}$
f. $\left\{\langle 0,2,1,-5\rangle + \left\{\vec{0}_4\right\}, \langle 2,-3,1,6\rangle + \left\{\vec{0}_4\right\}\right\}$
g. $\widetilde{T}_1(\langle 3,5,-2,4\rangle + W) = \langle 3,5,-2,4\rangle + \left\{\vec{0}_4\right\};$
$\widetilde{T}_1(\langle 1,2,7,-3\rangle + W) = \langle 1,2,7,-3\rangle + \left\{\vec{0}_4\right\};$
h. $\widetilde{T}_2(\langle 0,2,1,-5\rangle + V) = \langle 0,2,1,-5\rangle + \left\{\vec{0}_4\right\};$
$\widetilde{T}_2(\langle 2,-3,1,6\rangle + V) = \langle 2,-3,1,6\rangle + \left\{\vec{0}_4\right\}$

24. a. $\{\langle -3,-2,7,-4\rangle, \langle -2,13,-12,-2\rangle, \langle -2,3,-5,1\rangle, \langle -3,-5,6,-11\rangle\}$
b. $\{\langle -26,-17,14,0\rangle, \langle 3,-8,0,7\rangle\}$ c. $\{\langle -3,-2,7,-4\rangle + W\}$
d. $\{\langle -3,-2,7,-4\rangle + (V \cap W)\}$
e. $\{\langle -3,-5,6,-11\rangle + V\}$ f. $\{\langle -3,-5,6,-11\rangle + (V \cap W)\}$
g. $\widetilde{T}_1(\langle -3,-2,7,-4\rangle + W) = \langle -3,-2,7,-4\rangle + (V \cap W)$
h. $\widetilde{T}_2(\langle -3,-5,6,-11\rangle + V) = \langle -3,-5,6,-11\rangle + (V \cap W)$

25. a. $\{\langle -3,4,-1,4,6\rangle, \langle -6,8,5,15,-13\rangle, \langle 1,-2,0,-5,3\rangle, \langle 1,3,-2,7,2\rangle\}$
b. $\{\langle 3,-2,-5,0,4\rangle\}$
c. $\{\langle -3,4,-1,4,6\rangle + W, \langle -6,8,5,15,-13\rangle + W\}$
d. $\{\langle -3,4,-1,4,6\rangle + (V \cap W), \langle -6,8,5,15,-13\rangle + (V \cap W)\}$
e. $\{\langle 1,3,-2,7,2\rangle + V\}$ f. $\{\langle 1,3,-2,7,2\rangle + (V \cap W)\}$
g. $\widetilde{T}_1(\langle -3,4,-1,4,6\rangle + W) = \langle -3,4,-1,4,6\rangle + (V \cap W);$
$\widetilde{T}_1(\langle -6,8,5,15,-13\rangle + W) = \langle -6,8,5,15,-13\rangle + (V \cap W);$
h. $\widetilde{T}_2(\langle 1,3,-2,7,2\rangle + V) = \langle 1,3,-2,7,2\rangle + (V \cap W)$

26. a. $\{\langle -1,7,5,-6,6\rangle, \langle -1,-8,2,-4,2\rangle, \langle 1,0,3,-4,3\rangle, \langle 5,3,-2,7,-4\rangle, \langle -6,9,-2,0,0\rangle\}$
b. $\{\langle -17,31,-3,0,4\rangle, \langle -3,7,-1,2,0\rangle\}$
c. $\{\langle -1,7,5,-6,6\rangle + W, \langle -1,-8,2,-4,2\rangle + W\}$
d. $\{\langle -1,7,5,-6,6\rangle + (V \cap W), \langle -1,-8,2,-4,2\rangle + (V \cap W)\}$
e. $\{\langle -6,9,-2,0,0\rangle + V\}$ f. $\{\langle -6,9,-2,0,0\rangle + (V \cap W)\}$
g. $\widetilde{T}_1(\langle -1,7,5,-6,6\rangle + W) = \langle -1,7,5,-6,6\rangle + (V \cap W);$
$\widetilde{T}_1(\langle -1,-8,2,-4,2\rangle + W) = \langle -1,-8,2,-4,2\rangle + (V \cap W);$
h. $\widetilde{T}_2(\langle -6,9,-2,0,0\rangle + V) = \langle -6,9,-2,0,0\rangle + (V \cap W)$

Chapter Seven Exercises

7.1 Exercises

1. a. 3; b. –23; c. $-11/3$; d. $-5\sqrt{3}$; e. $4\ln 2 + 7\ln 3$; f. 1/2; g. –47 h. 148; i. 27/8; j. –29/3; k. 1968; l. $-70\ln 2 - 49\ln 5$
2. a. (i) ab; (ii) it is invertible ***if and only if*** both a and b are non-zero;
   (iii) $\frac{1}{ab}\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$.
   b. (i) $a^2 + b^2$; (ii) it is invertible ***if and only if*** either a or b is non-zero;
   (iii) $\frac{1}{a^2+b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.
   c. (i) $a^2 - b^2$; (ii) it is invertible ***if and only if*** $a \neq \pm b$; (iii) $\frac{1}{a^2-b^2}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$.
   d. (i) $2ab$; (ii) it is invertible ***if and only if*** both a and b are non-zero;
   (iii) $\frac{1}{2ab}\begin{bmatrix} b & -a \\ b & a \end{bmatrix} = \begin{bmatrix} \frac{1}{2a} & -\frac{1}{2b} \\ \frac{1}{2a} & \frac{1}{2b} \end{bmatrix}$.
   e. (i) $b - a$; (ii) it is invertible ***if and only if*** $a \neq b$; (iii) $\frac{1}{b-a}\begin{bmatrix} b & -a \\ -1 & 1 \end{bmatrix}$.
   f. (i) $2e^a$; (ii) it is always invertible; (i) $\frac{e^{-a}}{2}\begin{bmatrix} e^{-a} & -e^{-a} \\ e^{2a} & e^{2a} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-2a} & -e^{-2a} \\ e^{a} & e^{a} \end{bmatrix}$.
   g. (i) 1; (ii) it is always invertible; (iii) $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$.
   h. (i) 1; (ii) it is always invertible; (iii) $\begin{bmatrix} \cosh(a) & -\sinh(a) \\ -\sinh(a) & \cosh(a) \end{bmatrix}$
   i. (i) $\sin(\theta + \phi)$; (ii) it is invertible ***if and only if*** $\theta + \phi \neq n\pi$, where n is an integer;
   (iii) $\frac{1}{\sin(\theta+\phi)}\begin{bmatrix} \sin(\phi) & \sin(\theta) \\ -\cos(\phi) & \cos(\theta) \end{bmatrix}$
3. a. $(2,4,1,3)$; both have 3 inversions.
   b. $(5,3,2,4,1)$; both have 8 inversions. Notice that $\sigma = \sigma^{-1}$.
   c. $(5,3,6,1,4,2)$; both have 10 inversions.
   d. $(6,4,2,7,5,1,3)$; both have 14 inversions.
   e. $(5,7,3,8,4,1,6,2)$; both have 18 inversions.
   f. $(3,6,9,4,1,5,7,2,8)$; both have 16 inversions.
4. a. $(2,1,4,3)$; 2 inversions.
   b. $(2,3,5,4,1)$; 5 inversions.

c. $(4,1,2,5,6,3)$; 5 inversions.
d. $(6,3,4,2,5,1,7)$; 11 inversions.
e. $(6,2,3,5,1,7,8,4)$; 11 inversions.
f. $(5,8,1,9,6,2,7,4,3)$; 21 inversions.

5. a. 0; b. 0; c. −1. Only (c) is reversible.
6. a. 0; b. 0; c. −1.
7. $(c-a)(c-b)(b-a)$ (other factorizations are possible, up to ±1)
8. the permutation $\sigma = (n, n-1, \ldots, 3, 2, 1)$ will have $(n-1) + \cdots + 3 + 2 + 1 = (n-1)n/2$ inversions.

7.2 Exercises

1. a. (−) b. (+) c. (−) d. (+) e. (−) f. (−)
2. a. missing 2; (+) b. missing 4; (−) c. missing 3; (+)
   d. missing 5; (+) e. missing 2 and 5; (+) f. missing 7 and 4; (−)
3. a. column 2 is all zeroes; b. the third row is 4 times the first
4. a. −30; the matrix is upper triangular.
   b. 7/5; the matrix is upper triangular.
   c. −2640; the matrix is lower triangular.
   d. 60; the matrix is upper triangular.
   e. 3780; the matrix is upper triangular.
   f. −1/4; the matrix is lower triangular.
5. a. −560; b. 360; c. 720; d. −7/2.
6. a. −5; b. 5; c. −20; d. 1/3.
7. a. 42; b. 20; c. 10800; d. 40.
8. a. 9; b. 270; c. −252; d. 0. Hint: apply Type 3 column operations.
9. a. −70; b. 6; c. 480; d. −588
10. a. −321; b. 93; c. 2981; d. 403; e. 863; f. −1779; g. −182 h. −448
    i. −439; j. 9730; k. −29700; l. 214295
11. a. $det(A) = 76$; $det(B) = 345$; $det(C) = 421$.
    The three matrices are the same, except for their 2nd columns.
    The 2nd column of C is the sum of the 2nd column of A and the 2nd column of B.

7.3 Exercises

1. a. $det(A) = -34$ and $det(B) = 46$; b. $AB = \begin{bmatrix} 38 & 36 \\ 16 & -26 \end{bmatrix}$ and $det(AB) = -1564$.
   c. $-1564 = (-34)(46)$; d. $A + B = \begin{bmatrix} 11 & 4 \\ 4 & 5 \end{bmatrix}$ and $det(A+B) = 39$.
   e. $39 \neq -34 + 46$; f. $3B = \begin{bmatrix} 18 & -12 \\ 3 & 21 \end{bmatrix}$ and $det(3B) = 414$; g. $det(3B) = 9det(B)$.
2. a. $7\begin{vmatrix} -1 & 3 & -4 \\ 2 & -8 & 3 \\ 6 & 5 & 7 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -3 & 2 \\ -1 & 3 & -4 \\ 2 & -8 & 3 \end{vmatrix}$

b. first determinant is −149 and the other is −27; c. −1097

3. a. $-6\begin{vmatrix} -3 & 7 & -2 \\ 3 & 6 & 4 \\ -8 & -2 & 3 \end{vmatrix} - (-4)\begin{vmatrix} 5 & -3 & -2 \\ -1 & 3 & 4 \\ 2 & -8 & 3 \end{vmatrix}$

b. first determinant is −449 and the other is 168; c. 3366

4. a. $\begin{bmatrix} 4 & -2 & 3 & 8 \\ 9 & 0 & 17 & 28 \\ 2 & 3 & -2 & 3 \\ -3 & 0 & 9 & -5 \end{bmatrix}$ b. $\begin{bmatrix} 6 & 1 & 1 & 11 \\ 9 & 0 & 17 & 28 \\ 2 & 3 & -2 & 3 \\ -3 & 0 & 9 & -5 \end{bmatrix}$

c. $\begin{bmatrix} 6 & 1 & 1 & 11 \\ 9 & 0 & 17 & 28 \\ -16 & 0 & -5 & -30 \\ -3 & 0 & 9 & -5 \end{bmatrix}$ d. $-\begin{vmatrix} 9 & 17 & 28 \\ -16 & -5 & -30 \\ -3 & 9 & -5 \end{vmatrix}$

e. 1627

5. a. −255; b. 2452; c. −511; d. −1578
6. a. 56; b. −43; c. −42; d. −686.
7. 140
14. a. 1512; g. $r(x) = (x - a_1)(x - a_2)\cdots(x - a_k)$;
the bottom entry will be: $r(a_{k+1}) = (a_{k+1} - a_1)(a_{k+1} - a_2)\cdots(a_{k+1} - a_k)$

7.4 Exercises

1. a. $adj(A) = \begin{bmatrix} 4 & -5 \\ -1 & 3 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} \frac{4}{7} & -\frac{5}{7} \\ -\frac{1}{7} & \frac{3}{7} \end{bmatrix}$.

b. $adj(A) = \begin{bmatrix} 20 & 5 \\ -12 & -3 \end{bmatrix}$; A is not invertible.

c. $adj(A) = \begin{bmatrix} -4 & -7 & -2 \\ -10 & 5 & -5 \\ -1 & -13 & -23 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} \frac{4}{45} & \frac{7}{45} & \frac{2}{45} \\ \frac{2}{9} & -\frac{1}{9} & \frac{1}{9} \\ \frac{1}{45} & \frac{13}{45} & \frac{23}{45} \end{bmatrix}$.

d. $adj(A) = \begin{bmatrix} 14 & -6 & -31 \\ -7 & -24 & 2 \\ -35 & 15 & -17 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} -\frac{2}{27} & \frac{2}{63} & \frac{31}{189} \\ \frac{1}{27} & \frac{8}{63} & -\frac{2}{189} \\ \frac{5}{27} & -\frac{5}{63} & \frac{17}{189} \end{bmatrix}$.

e. $adj(A) = \begin{bmatrix} 183 & 85 & 74 & -62 \\ -534 & -338 & -379 & -44 \\ -63 & -63 & 42 & -63 \\ -339 & -388 & -362 & 53 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -\frac{61}{343} & -\frac{85}{1029} & -\frac{74}{1029} & \frac{62}{1029} \\ \frac{178}{343} & \frac{338}{1029} & \frac{379}{1029} & \frac{44}{1029} \\ \frac{3}{49} & \frac{3}{49} & -\frac{2}{49} & \frac{3}{49} \\ \frac{113}{343} & \frac{388}{1029} & \frac{362}{1029} & -\frac{53}{1029} \end{bmatrix}$

f. $adj(A) = \begin{bmatrix} 25 & 45 & 35 & -40 \\ -40 & -72 & -56 & 64 \\ 30 & 54 & 42 & -48 \\ -5 & -9 & -7 & 8 \end{bmatrix}$; A is not invertible.

2. a. $\langle x,y \rangle = \left\langle -\frac{31}{13}, -\frac{29}{13} \right\rangle$; b. $\langle x,y \rangle = \left\langle \frac{2}{59}, -\frac{92}{59} \right\rangle$
c. doesn't apply d. $\langle x,y \rangle = \left\langle \frac{3}{73}, -\frac{52}{73} \right\rangle$
e. $\langle x,y,z \rangle = \langle \frac{3}{4}, \frac{7}{4}, \frac{1}{2} \rangle$; f. doesn't apply
g. $\langle x,y,z \rangle = \left\langle \frac{209}{193}, \frac{66}{193}, \frac{367}{193} \right\rangle$; h. $\langle x,y,z \rangle = \left\langle -\frac{137}{83}, \frac{26}{83}, -\frac{15}{83} \right\rangle$
i. $\langle x,y,z,w \rangle = \left\langle \frac{164}{107}, -\frac{979}{107}, \frac{399}{107}, \frac{1029}{107} \right\rangle$; j. $\langle x,y,z,w \rangle = \langle 5,-8,6,0 \rangle$
k. $\langle x,y,z,w \rangle = \left\langle \frac{161}{44}, \frac{433}{88}, -\frac{247}{176}, -\frac{211}{44} \right\rangle$; l. $\langle x,y,z,w \rangle = \langle 1, -\frac{1}{2}, -\frac{1}{2}, 1 \rangle$

3. $b = \frac{3897}{6445}$; $c = -\frac{7733}{6445}$

4. $\langle 5,0,-2,7 \rangle$

5. $\begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$; $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; $\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. b. $adj(A) = \begin{bmatrix} -21 & -35 & 27 \\ 0 & 14 & -12 \\ 0 & 0 & -6 \end{bmatrix}$. It is also upper triangular.

7.5 Exercises

1. a. (i) $W_S(x) = 16\cos x \cos 3x \sin 2x - 9\cos x \cos 2x \sin 3x - 5\sin x \cos 2x \cos 3x$;
(ii) $W_S(\pi/4) = -8$, so S is linearly independent.
b. (i) $W_S(x) = -e^{2x}$; (ii) $W_S(0) = -1$, so S is linearly independent.
c. (i) $W_S(x) = -ne^{2kx}$; (ii) $W_S(x) = -n \neq 0$, so S is linearly independent.
d. (i) $W_S(x) = z(x)$; (ii) S is linearly dependent.

e. (i) $W_S(x) = z(x)$; (ii) S is linearly dependent.
f. (i) $W_S(x) = 18\cos^2 x\cos^2 2x + 18\cos^2 x\sin^2 2x + 18\sin^2 x\cos^2 2x + 18\sin^2 x\sin^2 2x$
(ii) b. $W_S(0) = 18$, so S is linearly independent.
g. (i) $W_S(x) = 12(\tan x)\sec^2(2x)\sec^2(3x)[3(\tan 3x) - 2(\tan 2x)]$
$+ 6\tan 2x\sec^2(3x)\sec^2(x)[(\tan x) - 3(\tan 3x)]$
$+ 4\tan 3x\sec^2(2x)\sec^2(x)[2(\tan 2x) - (\tan x)]$
(ii) $W_S(\pi/3) = 216$, so S is linearly independent.
h. (i) $W_S(x) = \frac{23}{160}x^{-\frac{3}{20}}$; (ii) $W_S(1) = \frac{23}{160}$, so S is linearly independent.
i. (i) $W_S(x) = \frac{1}{144\,000}x^{-\frac{283}{60}}$; (ii) $W_S(1) = \frac{1}{144\,000}$, so S is linearly independent.
j. (i) $W_S(x) = \left(-\frac{9}{16}\right)\frac{1}{[(x-1)(x-2)(x-3)(x-4)]^{3/2}}$; (ii) $W_S(5) \neq 0$, so S is linearly independent
(you can substitute any value of x that will not make the denominator zero).
k. (i) $W_S(x) = (5^x)(4^x)(3^x)(\ln 4 - \ln 3)(\ln 5 - \ln 3)(\ln 5 - \ln 4)$;
(ii) $W_S(0) = (\ln 4 - \ln 3)(\ln 5 - \ln 3)(\ln 5 - \ln 4) \neq 0$, so S is linearly independent.
l. (i) $W_S(x) = z(x)$; (ii) S is linearly dependent.

2. a. (i) $\{e^{k_1 x}, e^{k_2 x}, \ldots, e^{k_n x}\}$
(ii) $W_{S'}(x) = V(k_1, k_2, \ldots, k_n) \cdot e^{(k_1+k_2+\cdots+k_n)x}$
(iii) $W_{S'}(0) = V(k_1, k_2, \ldots, k_n) \neq 0$, since the k_i are distinct.
Thus, S is linearly independent.
b. (i) $\{b_1^x, b_2^x, \ldots, b_n^x\}$
(ii) $W_{S'}(x) = V(\ln(b_1), \ln(b_2), \ldots, \ln(b_n))b_1^x \cdot b_2^x \cdot \cdots \cdot b_n^x$
(iii) $W_{S'}(0) = V(\ln(b_1), \ln(b_2), \ldots, \ln(b_n)) \neq 0$, since the b_i are distinct.
Thus, S is linearly independent.
c. (i) $\{x^{k_1}, x^{k_2}, \ldots, x^{k_n}\}$
(ii) $W_{S'}(x) = V(k_1, k_2, \ldots, k_n)x^{k_1+k_2+\cdots+k_n-n(n+1)/2}$
(iii) $W_{S'}(0) = V(k_1, k_2, \ldots, k_n) \neq 0$, since the k_i are distinct.
Thus, S is linearly independent.
d. (i) $\{(x-k_1)^m, (x-k_2)^m, \ldots, (x-k_n)^m\}$
(ii) and (iii) There are two possibilities:
Case 1: If m is a positive integer and $n > m+1$, then the nth derivatives are all zero, so $W_{S'}(x) = z(x)$. Consequently, S will be dependent, since $\dim(\mathbb{R}^m) = m+1$, and S' contains $n > m+1$ vectors from $\mathbb{R}^m$.

Case 2: If m is not a positive integer, then m, $m-1$, …, $m-i$ are non-zero numbers for any positive integer i, and we get:

$$W_S(x) = \pm m \cdot m(m-1) \cdot m(m-1)(m-2) \cdot \cdots \cdot (m-n+2) \cdot (x-k_1)^{m+n-1}(x-k_2)^{m+n-1} \cdot \cdots \cdot (x-k_n)^{m+n-1} \cdot V(x-k_1, x-k_2, \ldots, x-k_n);$$

Note: the sign + or − depends on the remainder j when n is divided by 4, i.e. $n = 4i + j$, where i is a non-negative integer and $j = 0, 1, 2$, or 3, since we will need to perform row exchanges in order to bring the Wronskian matrix into a form similar to the Vandermonde matrix (note that the powers of $x - k_i$ are in decreasing rather than increasing order); the number of these exchanges depends on j; by letting x be any number bigger than k_n (where we assume the k_i are in increasing order), we get a non-zero value for $W_{S'}(x)$, so S is independent.

Chapter Eight Exercises

8.1 Exercises

1. Answers:
   a. $p(\lambda) = \lambda^2 + \lambda - 6$; $Eig(A,2) = Span(\{\langle -1,1\rangle\})$; $Eig(A,-3) = Span(\{\langle -2,1\rangle\})$. Each is 1-dimensional.
   b. $p(\lambda) = \lambda^2 - 8\lambda + 15$; $Eig(A,5) = Span(\{\langle 2,5\rangle\})$; $Eig(A,3) = Span(\{\langle 1,2\rangle\})$. Each is 1-dimensional.
   c. $p(\lambda) = \lambda^2 - 11\lambda - 12$; $Eig(A,-1) = Span(\{\langle -2,3\rangle\})$; $Eig(A,12) = Span(\{\langle 3,2\rangle\})$. Each is 1-dimensional.
   d. $p(\lambda) = \lambda^2 + 3\lambda - 10$; $Eig(A,2) = Span(\{\langle -4,3\rangle\})$; $Eig(A,-5) = Span(\{\langle -3,2\rangle\})$. Each is 1-dimensional.
   e. $p(\lambda) = \lambda^2 + 36$; since the eigenvalues are imaginary, there are no eigenvectors.
   f. $p(\lambda) = \lambda^2 - 15\lambda + 44$; $Eig(A,4) = Span(\{\langle 5,2\rangle\})$; $Eig(A,11) = Span(\{\langle 7,3\rangle\})$. Each is 1-dimensional.
   g. $p(\lambda) = (\lambda-5)(\lambda+2)(\lambda+4)$; $Eig(A,5) = Span(\{\langle 1,0,0\rangle\})$; $Eig(A,-2) = Span(\{\langle 4,-7,0\rangle\})$; $Eig(A,-4) = Span(\{\langle 2,27,18\rangle\})$. Each is 1-dimensional.
   h. $p(\lambda) = (\lambda-4)(\lambda-7)(\lambda+2)$; $Eig(A,4) = Span(\{\langle 6,2,1\rangle\})$; $Eig(A,7) = Span(\{\langle 0,-3,2\rangle\})$; $Eig(A,-2) = Span(\{\langle 0,0,1\rangle\})$. Each is 1-dimensional.
   i. $p(\lambda) = \lambda(\lambda+5)(\lambda-8)$; $Eig(A,0) = Span(\{\langle 3,-5,0\rangle\})$; $Eig(A,-5) = Span(\{\langle 1,0,0\rangle\})$; $Eig(A,8) = Span(\{\langle 69,-91,104\rangle\})$. Each is 1-dimensional.
   j. $p(\lambda) = (\lambda-3)^2(\lambda-2)(\lambda-4)$; $Eig(A,3) = Span(\{\langle 1,0,0,0\rangle,\langle 0,5,1,0\rangle\})$ is 2-dimensional; $Eig(A,2) = Span(\{\langle -3,1,0,0\rangle\})$; $Eig(A,4) = Span(\{\langle 27,-9,-2,2\rangle\})$; the other two are 1-dimensional.
   k. $p(\lambda) = (\lambda+2)^2(\lambda-3)^2$; $Eig(A,-2) = Span(\{\langle 5,4,0,0\rangle,\langle 0,0,-1,3\rangle\})$; $Eig(A,3) = Span(\{\langle 0,-1,2,0\rangle,\langle 0,0,0,1\rangle\})$. Each is 2-dimensional.
   l. $p(\lambda) = (\lambda-5)^3(\lambda+3)$; $Eig(A,5) = Span(\{\langle 0,0,1,0\rangle,\langle 0,0,0,1\rangle,\langle 8,7,0,0\rangle\})$ is 3-dimensional, and $Eig(A,-3) = Span(\{\langle 0,1,-3,2\rangle\})$ is 1-dimensional.
   m. $p(\lambda) = \lambda^2 - \lambda - 10/9$; $Eig(A,5/3) = Span(\{\langle -7,4\rangle\})$; $Eig(A,-2/3) = Span(\{\langle 1,-1\rangle\})$. Each is 1-dimensional.
   n. $p(\lambda) = (\lambda+1/3)(\lambda-4/3)(\lambda-2/3)$; $Eig(A,-1/3) = Span(\{\langle 1,0,0\rangle\})$; $Eig(A,4/3) = Span(\{\langle 1,1,0\rangle\})$; $Eig(A,2/3) = Span(\{\langle 3,1,2\rangle\})$. Each is 1-dimensional.
   o. $p(\lambda) = \lambda(\lambda-5/2)(\lambda+3/2)(\lambda-1/2)$; $Eig(A,5/2) = Span(\{\langle 1,0,0,0\rangle\})$; $Eig(A,0) = Span(\{\langle 7,5,0,0\rangle\})$; $Eig(A,-3/2) = Span(\{\langle 11,12,-4,0\rangle\})$; $Eig(A,1/2) = Span(\{\langle 57,34,10,-8\rangle\})$. Each is 1-dimensional.
2. Answers:
   a. A: $p(\lambda) = \lambda^2 - 36$; $Eig(A,6) = Span(\{\langle 3,2\rangle\})$; $Eig(A,-6) = Span(\{\langle -3,2\rangle\})$. B: the eigenvalues are imaginary: $\pm 6i$, so there are no eigenvectors.
   b. A: $p(\lambda) = (\lambda-3)^2(\lambda+2)$; $Eig(A,3) = Span(\{\langle 1,0,0\rangle,\langle 0,2,5\rangle\})$, 2-dimensional; $Eig(A,-2) = Span(\{\langle -3,1,0\rangle\})$, 1-dimensional. B: $p(\lambda) = (\lambda-3)^2(\lambda+2)$; $Eig(B,3) = Span(\{\langle 1,0,0\rangle\})$;

$Eig(B,-2) = Span(\{\langle -14,5,0\rangle\})$; both 1-dimensional.

c. A: $p(\lambda) = (\lambda+7)^2(\lambda-2)$; $Eig(A,-7) = Span(\{\langle 3,1,0\rangle, \langle 0,0,1\rangle\})$, 2-dimensional; $Eig(A,2) = Span(\{\langle 0,1,-2\rangle\})$, 1-dimensional.
B: $p(\lambda) = (\lambda+7)^2(\lambda-2)$; $Eig(B,-7) = Span(\{\langle 0,0,1\rangle\})$;
$Eig(B,2) = Span(\{\langle 0,1,2\rangle\})$; each is 1-dimensional.

d. A: $p(\lambda) = (\lambda-3)^2(\lambda+2)^2$; $Eig(A,-2) = Span(\{\langle 1,0,0,0\rangle\})$, 1-dimensional; $Eig(A,3) = Span(\{\langle -2,1,0,0\rangle, \langle 49,0,15,5\rangle\})$, 2-dimensional.
B: $p(\lambda) = (\lambda-3)^2(\lambda+2)^2$; $Eig(B,-2) = Span(\{\langle 1,0,0,0\rangle, \langle 0,7,5,0\rangle\})$;
$Eig(B,3) = Span(\{\langle -2,1,0,0\rangle, \langle 46,0,15,5\rangle\})$. Both are 2-dimensional.
C: $p(\lambda) = (\lambda-3)^2(\lambda+2)^2$; $Eig(C,-2) = Span(\{\langle 1,0,0,0\rangle, \langle 0,7,5,0\rangle\})$, 2-dimensional; $Eig(C,3) = Span(\{\langle -2,1,0,0\rangle\})$, 1-dimensional.

e. A: $p(\lambda) = (\lambda-3)(\lambda+2)^3$; $Eig(A,-2) = Span(\{\langle 1,0,0,0\rangle\})$;
$Eig(A,3) = Span(\{\langle -2,1,0,0\rangle\})$. Both are 1-dimensional.
B: $p(\lambda) = (\lambda-3)(\lambda+2)^3$; $Eig(B,3) = Span(\{\langle -4,1,0,0\rangle\})$, 1-dimensional;
$Eig(B,-2) = Span(\{\langle 1,0,0,0\rangle, \langle 0,2,5,0\rangle\})$, 2-dimensional.
C: $p(\lambda) = (\lambda-3)(\lambda+2)^3$; $Eig(C,3) = Span(\{\langle -4,1,0,0\rangle\})$, 1-dimensional;
$Eig(C,-2) = Span(\{\langle 1,0,0,0\rangle, \langle 0,2,5,0\rangle, \langle 0,-4,0,5\rangle\})$, 3-dimensional.

f. A: $p(\lambda) = (\lambda-3)^2(\lambda-1)^3$; $Eig(A,1) = Span(\{\langle 1,0,0,0,0\rangle, \langle 0,3,1,0,0\rangle\})$;
$Eig(A,3) = Span(\{\langle 2,1,0,0,0\rangle, \langle 0,0,3,1,0\rangle\})$. Both are 2-dimensional.
B: $p(\lambda) = (\lambda-3)^2(\lambda-1)^3$;
$Eig(B,1) = Span(\{\langle 1,0,0,0,0\rangle, \langle 0,3,1,0,0\rangle, \langle 0,0,0,-5,2\rangle\})$, 3-dimensional;
$Eig(B,3) = Span(\{\langle 2,1,0,0,0\rangle, \langle 0,0,3,1,0\rangle\})$, 2-dimensional.
C: $p(\lambda) = (\lambda-3)^2(\lambda-1)^3$;
$Eig(C,1) = Span(\{\langle 1,0,0,0,0\rangle, \langle 0,3,1,0,0\rangle, \langle 0,5,0,-5,2\rangle\})$, 3-dimensional;
$Eig(C,3) = Span(\{\langle 2,1,0,0,0\rangle\})$, 1-dimensional.

g. A: $p(\lambda) = \left(\lambda-\sqrt{3}\right)^2\left(\lambda-\sqrt{2}\right)$; $Eig\left(A,\sqrt{3}\right) = Span(\{\langle 1,1,0\rangle, \langle 0,0,1\rangle\})$, 2-dimensional;
$Eig\left(A,\sqrt{2}\right) = Span\left(\left\{\left\langle 0,\sqrt{3}-\sqrt{2},5\right\rangle\right\}\right)$, 1-dimensional.
B: $p(\lambda) = \left(\lambda-\sqrt{3}\right)^2\left(\lambda-\sqrt{2}\right)$; $Eig\left(B,\sqrt{3}\right) = Span(\{\langle 0,0,1\rangle\})$, 1-dimensional;
$Eig\left(B,\sqrt{2}\right) = Span\left(\left\{\left\langle 0,\sqrt{3}-\sqrt{2},5\right\rangle\right\}\right)$, 1-dimensional.

h. A: $p(\lambda) = (\lambda-3\pi^2)^2(\lambda-2\pi)$; $Eig(A,3\pi^2) = Span(\{\langle 0,0,1\rangle\})$, 1-dimensional;
$Eig(A,2\pi) = Span(\{\langle 0,3\pi-2,1\rangle\})$, 1-dimensional.
B: $p(\lambda) = (\lambda-3\pi^2)^2(\lambda-2\pi)$; $Eig(B,3\pi^2) = Span(\{\langle \pi,2,0\rangle, \langle 0,0,1\rangle\})$, 2-dimensional;
$Eig(B,2\pi) = Span(\{\langle 0,3\pi-2,1\rangle\})$, 1-dimensional.

3. $A^\top = \begin{bmatrix} -8 & 5 \\ -10 & 7 \end{bmatrix}$. We get the same characteristic polynomials and thus same eigenvalues.
However, for $A^\top$, $Eig(A^\top,-3) = Span(\{\langle 1,1\rangle\})$ and $Eig(A^\top,2) = Span(\{\langle 1,2\rangle\})$. These eigenspaces are different from the eigenspaces for A.

10. a. $\lambda^2 - 2\cos(\theta)\lambda + 1$; b. The discriminant is $-4\sin^2(\theta)$, which is negative unless $\sin(\theta) = 0$, which corresponds to $\theta = \pi n$. In this case, $\lambda = \cos(n\pi) = \pm 1$, and $R_\theta = \pm I$.

11. a. $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ b. rotate a vector $\vec{v}$ counterclockwise by θ then reflect this resulting

vector across the y-axis;

c. $\lambda^2 - 1$ d. the eigenvalues are always $\lambda = 1$ and $\lambda = -1$;

e. $Eig(A,-1) = Span(\{\langle \sin(\theta), 1 + \cos(\theta)\rangle\})$ and

$Eig(A,1) = Span(\{\langle \sin(\theta), -1 + \cos(\theta)\rangle\})$.

f. $Eig(A,-1) = Span(\{\langle \sin(\theta/2), \cos(\theta/2)\rangle\})$ and

$Eig(A,1) = Span(\{\langle \cos(\theta/2), -\sin(\theta/2)\rangle\})$.

h. they are orthogonal to each other!

i. $\begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix}$;

$Eig(A,-1) = Span(\{\langle -2,3\rangle\})$ and $Eig(A,1) = Span(\{\langle 3,2\rangle\})$.

j. Repeat (a) to (h) for the matrix B:

a. $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ b. reflect $\vec{v}$ across the x-axis, then rotate this resulting vector counterclockwise by θ.

c. $\lambda^2 - 1$ d. the eigenvalues are always $\lambda = 1$ and $\lambda = -1$;

e. $Eig(A,-1) = Span(\{\langle \sin(\theta), -1 - \cos(\theta)\rangle\})$ and

$Eig(A,1) = Span(\{\langle \sin(\theta), 1 - \cos(\theta)\rangle\})$.

f. $Eig(A,-1) = Span(\{\langle \sin(\theta/2), -\cos(\theta/2)\rangle\})$ and

$Eig(A,1) = Span(\{\langle \cos(\theta/2), \sin(\theta/2)\rangle\})$.

h. again, they are orthogonal to each other.

i. $\begin{bmatrix} -5/13 & 12/13 \\ 12/13 & 5/13 \end{bmatrix}$;

$Eig(A,-1) = Span(\{\langle -3,2\rangle\})$ and $Eig(A,1) = Span(\{\langle 2,3\rangle\})$.

12. b. $Eig(A_1 \oplus A_2, -5) = Span(\{\langle 0, 0, 1, 2, 1\rangle\})$;

$Eig(A_1 \oplus A_2, 3) = Span(\{\langle -2, 1, 0, 0, 0\rangle, \langle 0, 0, -1, 1, 0\rangle, \langle 0, 0, 1, 0, 1\rangle\})$;

$Eig(A_1 \oplus A_2, 7) = Span(\{\langle -5, 2, 0, 0, 0\rangle\})$

8.2 Exercises

1. For $\lambda = -5 : \{\langle 1,1,0\rangle\}$; for $\lambda = 3 : \{\langle 1,1,1\rangle\}$ and for $\lambda = 7 : \{\langle 0,-1,1\rangle\}$.
2. Hint: the exponent of p_1 can be $0, 1, 2, \ldots, n_1$.
3. Answers:
   a. 24 possibilities: $\pm\{1,2,3,5,6,9,10,15,18,30,45,90\}$; roots are: $\lambda = 5, 6, -3$
   b. 24 possibilities: $\pm\{1,2,3,4,6,8,9,12,18,24,36,72\}$; roots are: $\lambda = 6, -3, -4$
   c. 8 possibilities: $\pm\{1,3,5,15\}$; roots are: $\lambda = 5, 3 + \sqrt{6}, 3 - \sqrt{6}$.
4. Answers:
   a. $p(\lambda) = \lambda^3 - 9\lambda^2 + 23\lambda - 15 = (\lambda - 1)(\lambda - 3)(\lambda - 5)$; for $\lambda = 1 : \{\langle -1,0,1\rangle\}$, $dim = 1$;
   for $\lambda = 3 : \{\langle 1,0,1\rangle\}$, $dim = 1$; for $\lambda = 5 : \{\langle 0,1,0\rangle\}$, $dim = 1$.
   b. $p(\lambda) = \lambda^3 - 2\lambda^2 - 15\lambda + 36 = (\lambda - 3)^2(\lambda + 4)$;
   for $\lambda = -4 : \{\langle -7,0,1\rangle\}$, $dim = 1$; for $\lambda = 3 : \{\langle 0,0,1\rangle\}$, $dim = 1$.
   c. $p(\lambda) = \lambda^3 - 15\lambda^2 + 72\lambda - 112 = (\lambda - 7)(\lambda - 4)^2$;
   for $\lambda = 4 : \{\langle -1,1,0\rangle, \langle -1,0,1\rangle\}$, $dim = 2$; for $\lambda = 7 : \{\langle 1,1,1\rangle\}$, $dim = 1$.
   d. $p(\lambda) = \lambda^3 - 5\lambda^2 - 7\lambda + 35$; for $\lambda = -\sqrt{7} : \left\{\left\langle 1, -\sqrt{7} - 3, 0\right\rangle\right\}$, $dim = 1$;

for $\lambda = \sqrt{7}$: $\{\langle 1, \sqrt{7} - 3, 0\rangle\}$, $dim = 1$; for $\lambda = 5$: $\{\langle 0,0,1\rangle\}$, $dim = 1$.

e. $p(\lambda) = \lambda^3 - 3\lambda^2 - 10\lambda + 24 = (\lambda - 2)(\lambda - 4)(\lambda + 3)$; for $\lambda = -3$: $\{\langle 2,9,2\rangle\}$, $dim = 1$; for $\lambda = 2$: $\{\langle 36,42,31\rangle\}$, $dim = 1$; for $\lambda = 4$: $\{\langle 1,1,1\rangle\}$, $dim = 1$.

f. $p(\lambda) = \lambda^3 - 7/4\lambda^2 + 7/16\lambda + 15/64 = (\lambda + 1/4)(\lambda - 3/4)(\lambda - 5/4)$;
for $\lambda = -1/4$: $\{\langle 2,3,2\rangle\}$, $dim = 1$; for $\lambda = 3/4$: $\{\langle 1,1,1\rangle\}$, $dim = 1$;
for $\lambda = 5/4$: $\{\langle 4,4,3\rangle\}$, $dim = 1$.

g. $p(\lambda) = \lambda^3 - 13\lambda - 12 = (\lambda + 3)(\lambda + 1)(\lambda - 4)$; for $\lambda = -3$: $\{\langle -2,3,0\rangle\}$, $dim = 1$; for $\lambda = -1$: $\{\langle -1,0,1\rangle\}$, $dim = 1$; for $\lambda = 4$: $\{\langle 0,-1,1\rangle\}$, $dim = 1$.

h. $p(\lambda) = \lambda^3 - 15\lambda^2 + 72\lambda - 112 = (\lambda - 4)^2(\lambda - 7)$; for $\lambda = 4$: $\{\langle -2,5,0\rangle, \langle 4,0,5\rangle\}$, $dim = 2$; for $\lambda = 7$: $\{\langle 1,-1,1\rangle\}$, $dim = 1$.

i. $p(\lambda) = \lambda^3 - 15\lambda^2 + 72\lambda - 112 = (\lambda - 4)^2(\lambda - 7)$; (note: same as part (h)); for $\lambda = 4$: $\{\langle 2,-1,2\rangle\}$, $dim = 1$; for $\lambda = 7$: $\{\langle 1,-1,1\rangle\}$, $dim = 1$.

j. $p(\lambda) = \lambda^3 + \lambda^2 - 21\lambda - 45 = (\lambda - 5)(\lambda + 3)^2$; for $\lambda = -3$: $\{\langle -2,1,0\rangle, \langle 1,0,1\rangle\}$, $dim = 2$; for $\lambda = 5$: $\{\langle -4,2,1\rangle\}$, $dim = 1$.

k. $p(\lambda) = \lambda^3 - 5\lambda^2 - 32\lambda - 36 = (\lambda - 9)(\lambda + 2)^2$; for $\lambda = -2$: $\{\langle 1,1,0\rangle, \langle 1,0,1\rangle\}$, $dim = 2$; for $\lambda = 9$: $\{\langle -1,-4,2\rangle\}$, $dim = 1$.

l. $p(\lambda) = \lambda^3 - 7\lambda^2 - 5\lambda + 75 = (\lambda + 3)(\lambda - 5)^2$; for $\lambda = -3$: $\{\langle -1,-3,2\rangle\}$, $dim = 1$; for $\lambda = 5$: $\{\langle 2,3,0\rangle, \langle 2,0,3\rangle\}$, $dim = 2$;

m. $p(\lambda) = \lambda^3 + \frac{1}{3}\lambda^2 - \frac{40}{9}\lambda - \frac{112}{27} = (\lambda - 7/3)(\lambda + 4/3)^2$;
for $\lambda = -4/3$: $\{\langle -2,5,0\rangle, \langle 3,0,5\rangle\}$, $dim = 2$; for $\lambda = 7/3$: $\{\langle 1,1,2\rangle\}$, $dim = 1$.

n. $p(\lambda) = \lambda^3 + \frac{1}{4}\lambda^2 - \frac{33}{16}\lambda + \frac{63}{64} = (\lambda + 7/4)(\lambda - 3/4)^2$; for $\lambda = -7/4$: $\{\langle -2,-1,2\rangle\}$, $dim = 1$; for $\lambda = 3/4$: $\{\langle 3,1,0\rangle, \langle 3,0,5\rangle\}$, $dim = 2$;

o. $p(\lambda) = \lambda^3 - \frac{2}{5}\lambda^2 - \frac{3}{5}\lambda + \frac{36}{125} = (\lambda + 4/5)(\lambda - 3/5)^2$; for $\lambda = -4/5$: $\{\langle -1,-1,2\rangle\}$, $dim = 1$; for $\lambda = 3/5$: $\{\langle 2,1,0\rangle, \langle 3,0,5\rangle\}$, $dim = 2$;

p. $p(\lambda) = \lambda^4 - 25\lambda^2 - 3\lambda^3 + 75\lambda = (\lambda + 5)\lambda(\lambda - 3)(\lambda - 5)$;
for $\lambda = -5$: $\{\langle 3,0,-5,4\rangle\}$, $dim = 1$; for $\lambda = 0$: $\{\langle -4,0,0,3\rangle\}$, $dim = 1$; for $\lambda = 3$: $\{\langle 0,1,0,0\rangle\}$, $dim = 1$; for $\lambda = 5$: $\{\langle 3,0,5,4\rangle\}$, $dim = 1$.

q. $p(\lambda) = \lambda^4 - 98\lambda^2 + 2401 = (\lambda + 7)^2(\lambda - 7)^2$;
for $\lambda = -7$: $\{\langle -1,0,0,1\rangle, \langle 0,-1,1,0\rangle\}$, $dim = 2$;
for $\lambda = 7$: $\{\langle 1,0,0,1\rangle, \langle 0,1,1,0\rangle\}$, $dim = 2$.

r. $p(\lambda) = \lambda^4 - 116\lambda^2 + 1600 = (\lambda + 10)(\lambda + 4)(\lambda - 4)(\lambda - 10)$;
for $\lambda = -10$: $\{\langle 1,-1,1,-1\rangle\}$, $dim = 1$; for $\lambda = -4$: $\{\langle -1,-1,1,1\rangle\}$, $dim = 1$; for $\lambda = 4$: $\{\langle -1,1,1,-1\rangle\}$, $dim = 1$; for $\lambda = 10$: $\{\langle 1,1,1,1\rangle\}$, $dim = 1$.

s. $p(\lambda) = \lambda^4 - 7\lambda^3 + \lambda^2 + 63\lambda - 90 = (\lambda + 3)(\lambda - 2)(\lambda - 3)(\lambda - 5)$;
for $\lambda = -3$: $\{\langle 0,-1,0,1\rangle\}$, $dim = 1$; for $\lambda = 2$: $\{\langle 9,-1,3,2\rangle\}$, $dim = 1$; for $\lambda = 3$: $\langle 3,0,1,0\rangle$, $dim = 1$; for $\lambda = 5$: $\{\langle -1,-1,0,1\rangle\}$, $dim = 1$.

t. $p(\lambda) = \lambda^4 - 3\lambda^3 - 12\lambda^2 + 20\lambda + 48 = (\lambda + 2)^2(\lambda - 3)(\lambda - 4)$;
for $\lambda = -2$: $\{\langle 0,1,1,0\rangle, \langle -3,3,0,1\rangle\}$, $dim = 2$; for $\lambda = 3$: $\{\langle -2,1,-1,1\rangle\}$, $dim = 1$; for $\lambda = 4$: $\{\langle 5,-2,2,0\rangle\}$, $dim = 1$.

u. $p(\lambda) = \lambda^4 + 2\lambda^3 - 23\lambda^2 - 24\lambda + 144 = (\lambda + 4)^2(\lambda - 3)^2$;
for $\lambda = -4$: $\{\langle -2,1,-5,1\rangle\}$, $dim = 1$; for $\lambda = 3$: $\{\langle 1,0,2,0\rangle, \langle 1,1,0,1\rangle\}$, $dim = 2$.

v. $p(\lambda) = \lambda^4 - \lambda^3 - 18\lambda^2 + 52\lambda - 40 = (\lambda + 5)(\lambda - 2)^3$;
for $\lambda = -5$: $\{\langle -2,1,-3,1\rangle\}$, $dim = 1$;
for $\lambda = 2$: $\{\langle -3,2,0,0\rangle, \langle 1,0,2,0\rangle, \langle 3,0,0,2\rangle\}$, $dim = 3$.

w. $p(\lambda) = \lambda^4 - 5\lambda^3 + 6\lambda^2 + 4\lambda - 8 = (\lambda + 1)(\lambda - 2)^3$; for $\lambda = -1$: $\{\langle -2,1,-3,7\rangle\}$,

$dim = 1$; for $\lambda = 2$: $\{\langle -2,5,1,0\rangle, \langle -5,10,0,4\rangle\}$, $dim = 2$.

5. Answers:
   a. $p(\lambda) = \lambda^3 + \lambda^2 - 31\lambda + 46$; $Eig(A,-6.6758) = Span(\{\langle -0.60015,-0.8689,1\rangle\})$; $Eig(A, 1.7594) = Span(\{\langle 1.10664, 0.3865, 1\rangle\})$; $Eig(A, 3.9164) = Span(\{\langle -1.72078, 2.3395, 1\rangle\})$.
   b. $p(\lambda) = \lambda^3 + 8\lambda^2 + 7\lambda - 13$; $Eig(A,-6.6545) = Span(\{\langle -0.5515, 0.1185, 1\rangle\})$; $Eig(A,-2.2239) = Span(\{\langle 0.92536,-4.1326,1\rangle\})$; $Eig(A, 0.87843) = Span(\{\langle 1.9595, 0.680745, 1\rangle\})$.
   c. $p(\lambda) = \lambda^4 - 3\lambda^3 - 14\lambda^2 + 26\lambda + 10$;
   $Eig(A,-3.3149) = Span(\{\langle -0.158774, 0.156133,-0.072369,1\rangle\})$;
   $Eig(A,-0.33044) = Span(\{\langle -2.75815,-5.4277, 8.15926, 1\rangle\})$;
   $Eig(A, 1.9403) = Span(\{\langle 6.3174, 1.3771, 2.929, 1\rangle\})$;
   $Eig(A, 4.705) = Span(\{\langle 2.3495,-5.35553,-2.89094, 1\rangle\})$.

11. a. False. b. True. c. False. d. False. e. True. f. False. g. True.

8.3 Exercises

Note: the diagonal entries of D can be rearranged, as long as the corresponding eigenvectors are also located in the corresponding columns of C. As noted in the text, if eigenvalues appear in increasing order, and corresponding bases for eigenspaces also appear in the order that they are written when finding a basis for nullspaces by sightreading, then the matrices we obtain will be unique.

1. Answers:
   a. $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$; $C = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$
   b. $D = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$; $C = \begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix}$
   c. This matrix is not diagonalizable because the eigenvalues are imaginary.
   d. $D = \begin{bmatrix} -2/3 & 0 \\ 0 & 5/3 \end{bmatrix}$; $C = \begin{bmatrix} 1 & -7 \\ -1 & 4 \end{bmatrix}$
   e. $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 4 & 1 \\ 27 & -7 & 0 \\ 18 & 0 & 0 \end{bmatrix}$
   f. $D = \begin{bmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$
   g. $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$; $C = \begin{bmatrix} -3 & 1 & 0 & 27 \\ 1 & 0 & 5 & -9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

h. $D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

i. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$; $C = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

j. This matrix is not diagonalizable because there are only ***two*** linearly independent vectors, and this is a 3 × 3 matrix.

k. $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$; $C = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

l. $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 36 & 1 \\ 9 & 42 & 1 \\ 2 & 31 & 1 \end{bmatrix}$

m. $D = \begin{bmatrix} -1/4 & 0 & 0 \\ 0 & 3/4 & 0 \\ 0 & 0 & 5/4 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 4 \\ 2 & 1 & 3 \end{bmatrix}$

n. $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; $C = \begin{bmatrix} 2 & -1 & 0 \\ -3 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

o. $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$; $C = \begin{bmatrix} 4 & 2 & 1 \\ 0 & -5 & -1 \\ 5 & 0 & 1 \end{bmatrix}$

p. This matrix is not diagonalizable because there are only ***two*** linearly independent vectors, and this is a 3 × 3 matrix.

q. $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$; $C = \begin{bmatrix} -2 & 1 & -4 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

r. $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & -1 & -2 \end{bmatrix}$

s. $D = \begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$; $C = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

t. $D = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$; $C = \begin{bmatrix} 0 & -9 & 3 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & -3 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$

u. $D = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$; $C = \begin{bmatrix} 3 & -3 & 2 & -5 \\ 0 & 3 & -1 & 2 \\ 3 & 0 & 1 & -2 \\ -1 & 1 & -1 & 0 \end{bmatrix}$

v. This matrix is not diagonalizable because there are only ***three*** linearly independent vectors, and this is a 4×4 matrix.

w. $D = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$; $C = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 2 & 3 & -3 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

x. $D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$

2. Answers:
   a. Only matrix A is diagonalizable.
   b. Only matrix A is diagonalizable.
   c. Only matrix A is diagonalizable.
   d. Only matrix B is diagonalizable.
   e. Only matrix C is diagonalizable.
   f. Only matrix B is diagonalizable.
   g. Only matrix A is diagonalizable.
   h. Only matrix B is diagonalizable.

3. Answers:
   a. $\begin{bmatrix} -28381 & -37884 \\ 18942 & 25288 \end{bmatrix}$

b. $\begin{bmatrix} \frac{22003}{729} & \frac{22099}{729} \\ -\frac{12628}{729} & -\frac{12724}{729} \end{bmatrix}$

c. $\begin{bmatrix} 3125 & 1804 & -3167 \\ 0 & -32 & -1488 \\ 0 & 0 & -1024 \end{bmatrix}$

d. $\begin{bmatrix} 243 & -1477 & 1261 & 14472 \\ 0 & 32 & 1804 & -2996 \\ 0 & 0 & -3125 & -461 \\ 0 & 0 & 0 & 1024 \end{bmatrix}$

e. $\begin{bmatrix} -6493 & 3368 & 3368 \\ -23300 & 20175 & 3368 \\ 16564 & -16564 & 243 \end{bmatrix}$

f. $\begin{bmatrix} -5322 & -362 & 6708 \\ -4749 & -605 & 6378 \\ -5354 & -362 & 6740 \end{bmatrix}$

g. $\begin{bmatrix} 483 & 484 & 484 \\ -3801 & -2777 & -3801 \\ 3075 & 2050 & 3074 \end{bmatrix}$

h. $\begin{bmatrix} -77891 & -31566 & 63132 \\ 78915 & 32590 & -63132 \\ -78915 & -31566 & 64156 \end{bmatrix}$

i. $\begin{bmatrix} -59113 & 59081 & 59081 \\ -236324 & 236292 & 236324 \\ 118162 & -118162 & -118194 \end{bmatrix}$

j. $\begin{bmatrix} 0 & 0 & 0 & 16807 \\ 0 & 0 & 16807 & 0 \\ 0 & 16807 & 0 & 0 \\ 16807 & 0 & 0 & 0 \end{bmatrix}$

k. $\begin{bmatrix} 3125 & -1899 & -8646 & -1899 \\ 3368 & -518 & -10104 & -275 \\ 0 & -633 & 243 & -633 \\ -3368 & 550 & 10104 & 307 \end{bmatrix}$

l. $\begin{bmatrix} 7448 & 8030 & -8030 & -1650 \\ -3212 & -3519 & 3487 & 825 \\ 3212 & 3487 & -3519 & -825 \\ -1100 & -1375 & 1375 & 793 \end{bmatrix}$

4. Answers:

a. $\begin{bmatrix} 0 & 11664 \\ 5184 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 243 & 825 & -330 \\ 0 & -32 & 110 \\ 0 & 0 & 243 \end{bmatrix}$

c. $\begin{bmatrix} -16807 & 0 & 0 \\ -5613 & 32 & 0 \\ 11226 & -33678 & -16807 \end{bmatrix}$

d. $\begin{bmatrix} -32 & -550 & 770 & 220 \\ 0 & 243 & -385 & 1155 \\ 0 & 0 & -32 & 825 \\ 0 & 0 & 0 & 243 \end{bmatrix}$

e. $\begin{bmatrix} -32 & -1100 & 440 & -880 \\ 0 & 243 & -110 & 220 \\ 0 & 0 & -32 & 0 \\ 0 & 0 & 0 & -32 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 484 & -1452 & 4356 & 10890 \\ 0 & 243 & -726 & 2178 & 5445 \\ 0 & 0 & 1 & 726 & 1815 \\ 0 & 0 & 0 & 243 & 605 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

14. a. False b. False c. False d. True e. False f. True g. False h. True i. True j. False.

8.4 Exercises

1. a. $\langle\vec{v}\rangle_B = \langle -3, 7, -10\rangle$ and $\langle\vec{v}\rangle_{B'} = \langle 3, 8/3, 4/3\rangle$. b. The rrefs contained I_3 on the left side.

   c. $C_{B,B'} = \begin{bmatrix} -2 & 1 & 1 \\ \frac{4}{3} & 0 & -\frac{2}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix}$

2. a. $\langle\vec{v}\rangle_B = \langle 4, -3, 6, 33/2\rangle$ and $\langle\vec{v}\rangle_{B'} = \langle 5, 3, -7/2, 15/2\rangle$. b. The rrefs contained I_4 on the left side.

   c. $C_{B,B'} = \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & -1 \\ 0 & 2 & -6 & 3 \end{bmatrix}$.

3. a. $T(\vec{v}) = -\frac{49}{2}(\langle 0, -1, 1\rangle) + 15(\langle 1, -1, 1\rangle) - 23(\langle 1, 2, 1\rangle) = \langle -8, -73/2, -65/2\rangle$

   b. $[T] = \begin{bmatrix} -1 & 4 & -3 & 3 \\ -\frac{5}{2} & -\frac{7}{2} & \frac{13}{2} & 5 \\ -\frac{1}{2} & \frac{7}{2} & -\frac{1}{2} & 10 \end{bmatrix}$

4. a. $\begin{bmatrix} 4 & 3 & 1 \\ -3 & 1 & 0 \\ -5 & -2 & 4 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} -3 \\ 7 \\ -10 \end{bmatrix} = \begin{bmatrix} -1 \\ 16 \\ -39 \\ 13 \end{bmatrix}$. Decoding:

   $T(\vec{v}) = -1\langle 1, 0, 1, 2\rangle + 16\langle 0, 1, 1, -1\rangle - 39\langle 0, 0, 2, 1\rangle + 13\langle 0, 0, 0, -1\rangle = \langle -1, 16, -63, -70\rangle$.

   b. $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ -3 & 1 & 0 \\ -5 & -2 & 4 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 2 & -1 \\ -1 & -2 & 2 \\ -9 & 9 & 0 \\ 1 & 13 & -5 \end{bmatrix}$

5. a. $\begin{bmatrix} 6 & -3 & -1 \\ -2 & 1 & 0 \\ -7 & 2 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 7 \\ -10 \end{bmatrix} = \begin{bmatrix} -29 \\ 13 \\ -5 \end{bmatrix}$. Decoding, we get:

   $T(\vec{v}) = -29\langle 1, 0, -1\rangle + 13\langle 1, 1, 2\rangle - 5\langle 0, 1, 1\rangle = \langle -16, 8, 50\rangle$.

   b. $[T] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6 & -3 & -1 \\ -2 & 1 & 0 \\ -7 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & -\frac{5}{2} \\ -5 & 0 & 4 \\ -\frac{15}{2} & -\frac{9}{2} & \frac{19}{2} \end{bmatrix}$

6. a. $\begin{bmatrix} 7 & 3 & 1 \\ -1 & -4 & 0 \\ 3 & 5 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 26 \\ 8 \\ -17 \end{bmatrix}$ b. $\begin{bmatrix} 0 & \frac{3}{2} & -\frac{3}{2} \\ 6 & \frac{21}{2} & \frac{11}{2} \\ -7 & -\frac{31}{2} & -\frac{19}{2} \end{bmatrix}$

7. a. (i) $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\} = \{\langle -3,1,6,-5\rangle, \langle 4,2,-4,-4\rangle, \langle 1,4,7,3\rangle\}$
(ii) $B' = \{\langle 1,0,0,7\rangle, \langle 0,1,0,-8\rangle, \langle 0,0,1,4\rangle\}$
(iii) $\langle 18,4,-24,-2\rangle = -2\langle -3,1,6,-5\rangle + 3\langle 4,2,-4,-4\rangle$
(iv) $\langle 18,4,-24,-2\rangle = 18\langle 1,0,0,7\rangle + 4\langle 0,1,0,-8\rangle - 24\langle 0,0,1,4\rangle$
(v) $C_{B,B'} = \begin{bmatrix} -3 & 4 & 18 \\ 1 & 2 & 4 \\ 6 & -4 & -24 \end{bmatrix}$
(vi) $\begin{bmatrix} -3 & 4 & 18 \\ 1 & 2 & 4 \\ 6 & -4 & -24 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 18 \\ 4 \\ -24 \end{bmatrix}$

b. (i) $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{\langle -3,1,6,-5\rangle, \langle 4,2,-4,-4\rangle, \langle 1,4,7,3\rangle\}$
(ii) $B' = \{\langle 1,0,0,7\rangle, \langle 0,1,0,-8\rangle, \langle 0,0,1,4\rangle\}$
(iii) $\langle -10,-3,1,-42\rangle = 5\langle -3,1,6,-5\rangle + 2\langle 4,2,-4,-4\rangle - 3\langle 1,4,7,3\rangle$
(iv) $\langle -10,-3,1,-42\rangle = -10\langle 1,0,0,7\rangle - 3\langle 0,1,0,-8\rangle + 1\langle 0,0,1,4\rangle$
(v) $C_{B,B'} = \begin{bmatrix} -3 & 4 & 1 \\ 1 & 2 & 4 \\ 6 & -4 & 7 \end{bmatrix}$
(vi) $\begin{bmatrix} -3 & 4 & 1 \\ 1 & 2 & 4 \\ 6 & -4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -10 \\ -3 \\ 1 \end{bmatrix}$

c. (i) $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{\langle -3,12,5,2,-2\rangle, \langle 1,-4,4,3,-4\rangle, \langle 4,-16,-6,-4,18\rangle\}$
(ii) $B' = \{\langle 1,-4,0,0,3\rangle, \langle 0,0,1,0,5\rangle, \langle 0,0,0,1,-9\rangle\}$
For (iii) and (iv), there are no vectors from S which are not in B.
(v) $C_{B,B'} = \begin{bmatrix} -3 & 1 & 4 \\ 5 & 4 & -6 \\ 2 & 3 & -4 \end{bmatrix}$

d. (i) $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{\langle -3,-4,-2,9,1,1\rangle, \langle 1,2,4,9,11,-11\rangle, \langle 4,3,5,16,1,8\rangle\}$
(ii) $B' = \{\langle 1,0,0,3,-5,9\rangle, \langle 0,1,0,-7,2,-6\rangle, \langle 0,0,1,5,3,-2\rangle\}$
(iii)
$\langle -21,-36,-26,59,-45,79\rangle = 8\langle -3,-4,-2,9,1,1\rangle - 5\langle 1,2,4,9,11,-11\rangle + 2\langle 4,3,5,16,1,8\rangle$
$\langle -20,-37,-23,84,-43,88\rangle = 9\langle -3,-4,-2,9,1,1\rangle - 5\langle 1,2,4,9,11,-11\rangle + 3\langle 4,3,5,16,1,8\rangle$
(iv) $\langle -21,-36,-26,59,-45,79\rangle = -21\langle 1,0,0,3,-5,9\rangle - 36\langle 0,1,0,-7,2,-6\rangle$
$- 26\langle 0,0,1,5,3,-2\rangle$

$\langle -20,-37,-23,84,-43,88\rangle = -20\langle 1,0,0,3,-5,9\rangle - 37\langle 0,1,0,-7,2,-6\rangle$
$- 23\langle 0,0,1,5,3,-2\rangle$

(v) $C_{B,B'} = \begin{bmatrix} -3 & 1 & 4 \\ -4 & 2 & 3 \\ -2 & 4 & 5 \end{bmatrix}$

(vi) $\begin{bmatrix} -3 & 1 & 4 \\ -4 & 2 & 3 \\ -2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 8 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} -21 \\ -36 \\ -26 \end{bmatrix}$

$\begin{bmatrix} -3 & 1 & 4 \\ -4 & 2 & 3 \\ -2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -20 \\ -37 \\ -23 \end{bmatrix}$

e. (i) $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\} =$
$\{\langle -5,3,-3,2,-14,-4\rangle, \langle 3,-4,-7,-5,-21,7\rangle, \langle 2,-1,2,0,11,2\rangle, \langle -1,2,5,3,17,-8\rangle\}$
(ii) $B' = \{\langle 1,0,3,0,7,0\rangle, \langle 0,1,4,0,3,0\rangle, \langle 0,0,0,1,6,0\rangle, \langle 0,0,0,0,0,1\rangle\}$
(iii) $\langle -21,17,5,16,0,-26\rangle = 3\langle -5,3,-3,2,-14,-4\rangle - 2\langle 3,-4,-7,-5,-21,7\rangle$
(iv) $\langle -21,17,5,16,0,-26\rangle = -21\langle 1,0,3,0,7,0\rangle + 17\langle 0,1,4,0,3,0\rangle$
$+ 16\langle 0,0,0,1,6,0\rangle - 26\langle 0,0,0,0,0,1\rangle$

(v) $C_{B,B'} = \begin{bmatrix} -5 & 3 & 2 & -1 \\ 3 & -4 & -1 & 2 \\ 2 & -5 & 0 & 3 \\ -4 & 7 & 2 & -8 \end{bmatrix}$

(vi) $\begin{bmatrix} -5 & 3 & 2 & -1 \\ 3 & -4 & -1 & 2 \\ 2 & -5 & 0 & 3 \\ -4 & 7 & 2 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -21 \\ 17 \\ 16 \\ -26 \end{bmatrix}$

f. (i) $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\} =$
$\{\langle -4,-5,-1,3,7,1\rangle, \langle 2,3,-1,-1,-8,9\rangle, \langle -1,0,-4,2,2,1\rangle, \langle 3,2,6,-5,-4,-12\rangle\}$
(ii) $B' = \{\langle 1,0,4,0,0,9\rangle, \langle 0,1,-3,0,0,-6\rangle, \langle 0,0,0,1,0,7\rangle, \langle 0,0,0,0,1,-2\rangle\}$
(iii) $\langle -2,-1,-5,3,-10,29\rangle = 2\langle -4,-5,-1,3,7,1\rangle + 3\langle 2,3,-1,-1,-8,9\rangle$
(iv) $\langle -2,-1,-5,3,-10,29\rangle = -2\langle 1,0,4,0,0,9\rangle - \langle 0,1,-3,0,0,-6\rangle$
$+ 3\langle 0,0,0,1,0,7\rangle - 10\langle 0,0,0,0,1,-2\rangle$

(v) $C_{B,B'} = \begin{bmatrix} -4 & 2 & -1 & 3 \\ -5 & 3 & 0 & 2 \\ 3 & -1 & 2 & -5 \\ 7 & -8 & 2 & -4 \end{bmatrix}$

(vi) $\begin{bmatrix} -4 & 2 & -1 & 3 \\ -5 & 3 & 0 & 2 \\ 3 & -1 & 2 & -5 \\ 7 & -8 & 2 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \\ -10 \end{bmatrix}$

g. (i) $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{\langle -3,1,4,-21,-20\rangle, \langle -4,2,3,-36,-37\rangle, \langle -2,4,5,-26,-23\rangle\}$

(ii) $B^{/} = \{\langle 1,0,0,8,9\rangle, \langle 0,1,0,-5,-5\rangle, \langle 0,0,1,2,3\rangle\}$

(iii) $\langle 9,9,16,59,84\rangle = 3\langle -3,1,4,-21,-20\rangle - 7\langle -4,2,3,-36,-37\rangle + 5\langle -2,4,5,-26,-23\rangle$

$\langle 1,11,1,-45,-43\rangle = -5\langle -3,1,4,-21,-20\rangle + 2\langle -4,2,3,-36,-37\rangle + 3\langle -2,4,5,-26,-23\rangle$

$\langle 1,-11,8,79,88\rangle = 9\langle -3,1,4,-21,-20\rangle - 6\langle -4,2,3,-36,-37\rangle - 2\langle -2,4,5,-26,-23\rangle$

(iv) $\langle 9,9,16,59,84\rangle = 9\langle 1,0,0,8,9\rangle + 9\langle 0,1,0,-5,-5\rangle + 16\langle 0,0,1,2,3\rangle$

$\langle 1,11,1,-45,-43\rangle = \langle 1,0,0,8,9\rangle + 11\langle 0,1,0,-5,-5\rangle + \langle 0,0,1,2,3\rangle$

$\langle 1,-11,8,79,88\rangle = \langle 1,0,0,8,9\rangle - 11\langle 0,1,0,-5,-5\rangle + 8\langle 0,0,1,2,3\rangle$

(v) $C_{B,B^{/}} = \begin{bmatrix} -3 & -4 & -2 \\ 1 & 2 & 4 \\ 4 & 3 & 5 \end{bmatrix}$

(vi) $\begin{bmatrix} -3 & -4 & -2 \\ 1 & 2 & 4 \\ 4 & 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ -7 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 16 \end{bmatrix}$

$\begin{bmatrix} -3 & -4 & -2 \\ 1 & 2 & 4 \\ 4 & 3 & 5 \end{bmatrix}\begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -3 & -4 & -2 \\ 1 & 2 & 4 \\ 4 & 3 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 8 \end{bmatrix}$

h. (i) $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} =$
$\{\langle -4,-5,3,19,2,-8\rangle, \langle -8,-1,2,-28,3,-26\rangle, \langle 2,2,-1,-5,0,15\rangle, \langle 7,3,-4,5,-5,-8\rangle\}$

(ii) $B^{/} = \{\langle 1,0,0,5,0,8\rangle, \langle 0,1,0,-6,0,3\rangle, \langle 0,0,1,3,0,7\rangle, \langle 0,0,0,0,1,9\rangle\}$

(iii) $\langle 8,7,-10,-32,-8,-57\rangle = 5\langle -4,-5,3,19,2,-8\rangle + 4\langle -8,-1,2,-28,3,-26\rangle + 9\langle 2,2,-1,-5,0,15\rangle + 6\langle 7,3,-4,5,-5,-8\rangle$

(iv) $\langle 8,7,-10,-32,-8,-57\rangle = 8\langle 1,0,0,5,0,8\rangle + 7\langle 0,1,0,-6,0,3\rangle - 10\langle 0,0,1,3,0,7\rangle - 8\langle 0,0,0,0,1,9\rangle$

(v) $C_{B,B^{/}} = \begin{bmatrix} -4 & -8 & 2 & 7 \\ -5 & -1 & 2 & 3 \\ 3 & 2 & -1 & -4 \\ 2 & 3 & 0 & -5 \end{bmatrix}$

(vi) $\begin{bmatrix} -4 & -8 & 2 & 7 \\ -5 & -1 & 2 & 3 \\ 3 & 2 & -1 & -4 \\ 2 & 3 & 0 & -5 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ -10 \\ -8 \end{bmatrix}$

8.5 Exercises

1. a. $\langle \vec{v} \rangle_B = \langle -11, 6, 9 \rangle$ and $\langle \vec{v} \rangle_{B'} = \langle 3, 0, 2 \rangle$. c. $C_{B,B'} = \begin{bmatrix} -1 & -\frac{4}{3} & 0 \\ 1 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$.

2. a. $\langle \vec{v} \rangle_B = \langle 2, 4, 1, -5 \rangle$ and $\langle \vec{v} \rangle_{B'} = \langle 5, -3, 16, -70 \rangle$.

   c. $C_{B,B'} = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 1 & -1 & -1 & 0 \\ 4 & 6 & 14 & 6 \\ -19 & -27 & -64 & -28 \end{bmatrix}$.

3. a. $T(\vec{v}) = 27(x - x^2) - 49(1 + x) + 38(2 - x^2) = 27 - 22x - 65x^2$

   b. $[T]_{S,S'} = \begin{bmatrix} 2 & 17 & 26 & 18 \\ \frac{7}{2} & \frac{29}{2} & \frac{19}{2} & 17 \\ \frac{3}{2} & \frac{11}{2} & -\frac{15}{2} & 13 \end{bmatrix}$.

4. a. $T(\vec{v}) = 27 + 36x - 169x^2 + 144x^3$. b. $[T]_{S,S'} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & -2 & -6 \\ -14 & -6 & 19 \\ 10 & 8 & -16 \end{bmatrix}$

5. a. $T(\vec{v}) = -21 + 29x - 29x^2$. b. $[B]_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $[B]_S^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$.

   c. $[T]_S = \begin{bmatrix} -3 & 1 & 4 \\ 2 & 2 & -3 \\ -3 & 2 & 3 \end{bmatrix}$. e. $det(T) = -7$. f. Yes. $[T^{-1}]_B = \begin{bmatrix} \frac{5}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{3}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{1}{7} & -\frac{3}{7} & -\frac{1}{7} \end{bmatrix}$.

6. a. $T(\vec{v}) = 116 - 63x + 27x^2 + 19x^3$.

b. $[B]_S = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ and $[B]_S^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ \frac{1}{2} & \frac{5}{2} & \frac{3}{2} & 3 \end{bmatrix}$.

c. $[T]_S = \begin{bmatrix} -\frac{13}{2} & -\frac{99}{2} & -\frac{51}{2} & -51 \\ \frac{9}{2} & \frac{57}{2} & \frac{33}{2} & 33 \\ -2 & -11 & -5 & -12 \\ -\frac{3}{2} & -\frac{19}{2} & -\frac{13}{2} & -12 \end{bmatrix}$ d. $det(T) = 0$. e. No.

7. a. $[D]_B = \begin{bmatrix} 2 & 3 & -1 \\ -2 & -3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ b. $[D]_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ c. $det(T) = 0$. d. No.

8. a. $[D]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ -4 & \frac{9}{2} & -\frac{1}{2} & 0 \end{bmatrix}$ b. $[D]_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ c. $det(T) = 0$. d. No.

9. a. The members of $B^{/}$ are non-zero, non-parallel linear combinations of $\sin(x)$ and $\cos(x)$.

b. $\begin{bmatrix} \sqrt{3} & -1 \\ -1 & \sqrt{3} \end{bmatrix}$ c. $[T]_B = \begin{bmatrix} 5 + 3\sqrt{3} & -4\sqrt{3} - 6 \\ 4\sqrt{3} + 6 & -11 - 3\sqrt{3} \end{bmatrix}$ d. $det(T) = 2$.

e. Yes; $[T^{-1}]_B = \begin{bmatrix} -\frac{11}{2} - \frac{3}{2}\sqrt{3} & 2\sqrt{3} + 3 \\ -2\sqrt{3} - 3 & \frac{5}{2} + \frac{3}{2}\sqrt{3} \end{bmatrix}$ f. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

g. $det(D) = 1$. h. Yes. $[D^{-1}]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

10. a. $[D]_B = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$. b. $det(D) = -8$ c. Yes. $[D^{-1}]_B = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$.

8.6 Exercises

1. a. $[T]_S = \begin{bmatrix} 0 & -5 & -14 \\ 0 & 3 & -10 \\ 0 & 0 & 14 \end{bmatrix}$ b. $det(T) = 0$; c. $p(\lambda) = \lambda(\lambda - 3)(\lambda - 14)$ d. $\lambda = 0, 3, 14$

   e. $Eig(T, 0) = Span(\{1\})$; $Eig(T, 3) = Span(\{-5 + 3x\})$;
   $Eig(T, 14) = Span(\{52 + 70x - 77x^2\})$

   f. $[T]_B = Diag(0, 3, 14)$, where $B = \{1, -5 + 3x, 52 + 70x - 77x^2\}$.

2. a. $[T]_S = \begin{bmatrix} 4 & 5 & -8 & 0 \\ 0 & 6 & 14 & -24 \\ 0 & 0 & 14 & 27 \\ 0 & 0 & 0 & 28 \end{bmatrix}$ b. $det(T) = 9408$;

   c. $p(\lambda) = (\lambda - 4)(\lambda - 6)(\lambda - 14)(\lambda - 28)$; d. $\lambda = 4, 6, 14, 28$

   e. $Eig(T, 4) = Span(\{1\})$; $Eig(T, 6) = Span(\{5 + 2x\})$;
   $Eig(T, 14) = Span(\{3 + 70x + 40x^2\})$;
   $Eig(T, 28) = Span(\{-757 + 168x + 2376x^2 + 1232x^3\})$;

   f. $[T]_B = Diag(4, 6, 14, 28)$, where
   $B = \{1, 5 + 2x, 3 + 70x + 40x^2, -757 + 168x + 2376x^2 + 1232x^3\}$.

3. a. $[T]_S = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$ b. $det(T) = 25$; c. $p(\lambda) = \lambda^2 + 25$

   d. The eigenvalues are imaginary, so . . . e. there are no eigenvectors for T, and consequently, . . .

   f. T is not diagonalizable.

4. a. $[D]_S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ b. $det(D) = -10$; c. $p(\lambda) = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

   d. $\lambda = -1, 2, 5$; f. $Eig(D, -1) = Span(\{e^{-x}\})$;
   $Eig(D, 2) = Span(\{e^{2x}\})$; $Eig(D, 5) = Span(\{e^{5x}\})$;

   g. $[D]_S$ is already diagonal, so it is diagonalizable.

5. a. $[D]_S = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ b. $det(D) = 27$; c. $p(\lambda) = (\lambda - 3)^3$ d. $\lambda = 3$

   e. $Eig(D, 3) = Span(\{e^{3x}\})$ f. D is not diagonalizable.

6. a. $\lambda = -1$ for both $\sin(x)$ and $\cos(x)$. b. The eigenvalue of $e^{\lambda x}$ is λ^2. c. $e^{\sqrt{\mu}x}$ d. $-\lambda^2$

   e. It has the same eigenvalue, $-\lambda^2$. f. The common eigenvalue is 1.

7. $[T]_B = Diag(5, -1, 4)$, where $B = \{1, 1 - 3x, 3 + 6x + 5x^2\}$

8. $$[T]_S = \begin{bmatrix} 3 & 2 & 1 \\ -5 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{12} & -\frac{1}{4} & \frac{1}{6} \\ \frac{5}{12} & \frac{1}{4} & \frac{1}{6} \\ \frac{5}{12} & \frac{1}{4} & -\frac{5}{6} \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{67}{12} & -\frac{23}{4} & -\frac{17}{6} \\ \frac{5}{12} & \frac{17}{4} & -\frac{5}{6} \\ -\frac{25}{6} & -\frac{5}{2} & \frac{4}{3} \end{bmatrix}.$$

8.7 Exercises

We provide the answers for e^{tA}. To get e^A, just replace t with 1.

1. $$\begin{bmatrix} -8e^{2t} + 9e^{-5t} & -12e^{2t} + 12e^{-5t} \\ 6e^{2t} - 6e^{-5t} & 9e^{2t} - 8e^{-5t} \end{bmatrix}$$

2. $$\begin{bmatrix} -\frac{4}{3}e^{-\frac{2}{3}t} + \frac{7}{3}e^{\frac{5}{3}t} & -\frac{7}{3}e^{-\frac{2}{3}t} + \frac{7}{3}e^{\frac{5}{3}t} \\ \frac{4}{3}e^{-\frac{2}{3}t} - \frac{4}{3}e^{\frac{5}{3}t} & \frac{7}{3}e^{-\frac{2}{3}t} - \frac{4}{3}e^{\frac{5}{3}t} \end{bmatrix}$$

3. $$\begin{bmatrix} e^{5t} & -\frac{4}{7}e^{-2t} + \frac{4}{7}e^{5t} & \frac{6}{7}e^{-2t} + \frac{1}{9}e^{-4t} - \frac{61}{63}e^{5t} \\ 0 & e^{-2t} & -\frac{3}{2}e^{-2t} + \frac{3}{2}e^{-4t} \\ 0 & 0 & e^{-4t} \end{bmatrix}$$

4. $$\begin{bmatrix} e^{3t} & -3e^{2t} + 3e^{3t} & 15e^{2t} - 15e^{3t} & \frac{3}{2}e^{2t} - 15e^{3t} + \frac{27}{2}e^{4t} \\ 0 & e^{2t} & -5e^{2t} + 5e^{3t} & -\frac{1}{2}e^{2t} + 5e^{3t} - \frac{9}{2}e^{4t} \\ 0 & 0 & e^{3t} & e^{3t} - e^{4t} \\ 0 & 0 & 0 & e^{4t} \end{bmatrix}$$

5. $$\begin{bmatrix} -e^{3t} + 2e^{-5t} & e^{3t} - e^{-5t} & e^{3t} - e^{-5t} \\ -e^{3t} + 2e^{-5t} - e^{7t} & e^{3t} - e^{-5t} + e^{7t} & e^{3t} - e^{-5t} \\ -e^{3t} + e^{7t} & e^{3t} - e^{7t} & e^{3t} \end{bmatrix}$$

6. $$\begin{bmatrix} \frac{36}{5}e^{2t} - \frac{22}{35}e^{-3t} - \frac{39}{7}e^{4t} & \frac{2}{7}e^{-3t} - \frac{2}{7}e^{4t} & -\frac{36}{5}e^{2t} + \frac{12}{35}e^{-3t} + \frac{48}{7}e^{4t} \\ \frac{42}{5}e^{2t} - \frac{99}{35}e^{-3t} - \frac{39}{7}e^{4t} & \frac{9}{7}e^{-3t} - \frac{2}{7}e^{4t} & -\frac{42}{5}e^{2t} + \frac{54}{35}e^{-3t} + \frac{48}{7}e^{4t} \\ \frac{31}{5}e^{2t} - \frac{22}{35}e^{-3t} - \frac{39}{7}e^{4t} & \frac{2}{7}e^{-3t} - \frac{2}{7}e^{4t} & -\frac{31}{5}e^{2t} + \frac{12}{35}e^{-3t} + \frac{48}{7}e^{4t} \end{bmatrix}$$

7. $\begin{bmatrix} 3e^{-t}-2e^{-3t} & 2e^{-t}-2e^{-3t} & 2e^{-t}-2e^{-3t} \\ 3e^{-3t}-3e^{4t} & 3e^{-3t}-2e^{4t} & 3e^{-3t}-3e^{4t} \\ -3e^{-t}+3e^{4t} & -2e^{-t}+2e^{4t} & -2e^{-t}+3e^{4t} \end{bmatrix}$

8. $\begin{bmatrix} 6e^{4t}-5e^{7t} & 2e^{4t}-2e^{7t} & -4e^{4t}+4e^{7t} \\ -5e^{4t}+5e^{7t} & -e^{4t}+2e^{7t} & 4e^{4t}-4e^{7t} \\ 5e^{4t}-5e^{7t} & 2e^{4t}-2e^{7t} & -3e^{4t}+4e^{7t} \end{bmatrix}$

9. $\begin{bmatrix} 2e^{-2t}-e^{9t} & -e^{-2t}+e^{9t} & -e^{-2t}+e^{9t} \\ 4e^{-2t}-4e^{9t} & -3e^{-2t}+4e^{9t} & -4e^{-2t}+4e^{9t} \\ -2e^{-2t}+2e^{9t} & 2e^{-2t}-2e^{9t} & 3e^{-2t}-2e^{9t} \end{bmatrix}$

10. $\begin{bmatrix} \frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & 0 & 0 & -\frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} \\ 0 & \frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & -\frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & 0 \\ 0 & -\frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & \frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & 0 \\ -\frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} & 0 & 0 & \frac{1}{2}e^{-7t}+\frac{1}{2}e^{7t} \end{bmatrix}$

11. $\begin{bmatrix} e^{5t} & 9e^{2t}-9e^{3t} & 3e^{3t}-3e^{5t} & 9e^{2t}-9e^{3t} \\ -e^{-3t}+e^{5t} & -e^{2t}+2e^{-3t} & 3e^{-3t}-3e^{5t} & -e^{2t}+e^{-3t} \\ 0 & 3e^{2t}-3e^{3t} & e^{3t} & 3e^{2t}-3e^{3t} \\ e^{-3t}-e^{5t} & 2e^{2t}-2e^{-3t} & -3e^{-3t}+3e^{5t} & 2e^{2t}-e^{-3t} \end{bmatrix}$

12. $\begin{bmatrix} -12e^{-2t}+8e^{3t}+5e^{4t} & -15e^{-2t}+10e^{3t}+5e^{4t} & 15e^{-2t}-10e^{3t}-5e^{4t} & 6e^{-2t}-6e^{3t} \\ 6e^{-2t}-4e^{3t}-2e^{4t} & 8e^{-2t}-5e^{3t}-2e^{4t} & -7e^{-2t}+5e^{3t}+2e^{4t} & -3e^{-2t}+3e^{3t} \\ -6e^{-2t}+4e^{3t}+2e^{4t} & -7e^{-2t}+5e^{3t}+2e^{4t} & 8e^{-2t}-5e^{3t}-2e^{4t} & 3e^{-2t}-3e^{3t} \\ 4e^{-2t}-4e^{3t} & 5e^{-2t}-5e^{3t} & -5e^{-2t}+5e^{3t} & -2e^{-2t}+3e^{3t} \end{bmatrix}$

13. $\begin{bmatrix} -3e^{2t}+4e^{-5t} & -6e^{2t}+6e^{-5t} & 2e^{2t}-2e^{-5t} & 6e^{2t}-6e^{-5t} \\ 2e^{2t}-2e^{-5t} & 4e^{2t}-3e^{-5t} & -e^{2t}+e^{-5t} & -3e^{2t}+3e^{-5t} \\ -6e^{2t}+6e^{-5t} & -9e^{2t}+9e^{-5t} & 4e^{2t}-3e^{-5t} & 9e^{2t}-9e^{-5t} \\ 2e^{2t}-2e^{-5t} & 3e^{2t}-3e^{-5t} & -e^{2t}+e^{-5t} & -2e^{2t}+3e^{-5t} \end{bmatrix}$

14. $\begin{bmatrix} \frac{1}{2}e^{4t}+\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}e^{4t}-\frac{1}{2} \\ 0 & \frac{1}{2}e^{4t}+\frac{1}{2} & 0 & \frac{1}{2}e^{4t}-\frac{1}{2} & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \\ 0 & \frac{1}{2}e^{4t}-\frac{1}{2} & 0 & \frac{1}{2}e^{4t}+\frac{1}{2} & 0 \\ \frac{1}{2}e^{4t}-\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}e^{4t}+\frac{1}{2} \end{bmatrix}$

15. $$\begin{bmatrix} \frac{1}{2}e^{-6t} + \frac{1}{2}e^{6t} & -\frac{3}{4}e^{-6t} + \frac{3}{4}e^{6t} \\ -\frac{1}{3}e^{-6t} + \frac{1}{3}e^{6t} & \frac{1}{2}e^{-6t} + \frac{1}{2}e^{6t} \end{bmatrix}$$

16. $$\begin{bmatrix} e^{3t} & -3e^{-2t} + 3e^{3t} & \frac{6}{5}e^{-2t} - \frac{6}{5}e^{3t} \\ 0 & e^{-2t} & -\frac{2}{5}e^{-2t} + \frac{2}{5}e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}$$

17. $$\begin{bmatrix} e^{-7t} & 0 & 0 \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-7t} & e^{2t} & 0 \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-7t} & -2e^{2t} + 2e^{-7t} & e^{-7t} \end{bmatrix}$$

18. $$\begin{bmatrix} e^{-2t} & 2e^{-2t} - 2e^{3t} & -\frac{14}{5}e^{-2t} + \frac{14}{5}e^{3t} & -\frac{4}{5}e^{-2t} + \frac{4}{5}e^{3t} \\ 0 & e^{3t} & \frac{7}{5}e^{-2t} - \frac{7}{5}e^{3t} & -\frac{21}{5}e^{-2t} + \frac{21}{5}e^{3t} \\ 0 & 0 & e^{-2t} & -3e^{-2t} + 3e^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

19. $$\begin{bmatrix} e^{-2t} & 4e^{-2t} - 4e^{3t} & -\frac{8}{5}e^{-2t} + \frac{8}{5}e^{3t} & \frac{16}{5}e^{-2t} - \frac{16}{5}e^{3t} \\ 0 & e^{3t} & \frac{2}{5}e^{-2t} - \frac{2}{5}e^{3t} & -\frac{4}{5}e^{-2t} + \frac{4}{5}e^{3t} \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{bmatrix}$$

20. $$\begin{bmatrix} e^{t} & -2e^{t} + 2e^{3t} & 6e^{t} - 6e^{3t} & -18e^{t} + 18e^{3t} & -45e^{t} + 45e^{3t} \\ 0 & e^{3t} & 3e^{t} - 3e^{3t} & -9e^{t} + 9e^{3t} & -\frac{45}{2}e^{t} + \frac{45}{2}e^{3t} \\ 0 & 0 & e^{t} & -3e^{t} + 3e^{3t} & -\frac{15}{2}e^{t} + \frac{15}{2}e^{3t} \\ 0 & 0 & 0 & e^{3t} & -\frac{5}{2}e^{t} + \frac{5}{2}e^{3t} \\ 0 & 0 & 0 & 0 & e^{t} \end{bmatrix}$$

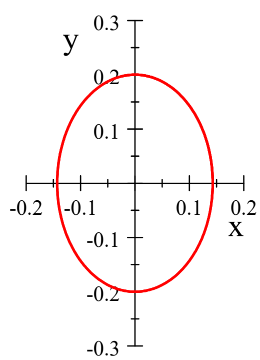
Chapter Nine Exercises

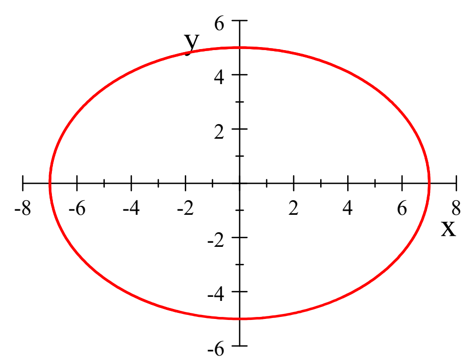
9.1 Exercises

1. a. -46; b. $-22/5$; c. 16; d. -276; e. 22; f. -72; g. 38; h. $-10,892$
2. a. $1/2$; b. 0
3. b. $r(x) = (x+1)(x-1)(x-2)(x-4)$ or any scalar multiple thereof.
4. No.
5. No.
6. Yes.
7. a. $-\pi/4$
14. b. removable discontinuity
19. a. Further hint: since the series $\sum a_n$ converges, the terms a_n must converge to 0, so therefore if n is large enough, $|a_n| < 1$. c. (use geometric series formula) $1/5$; d. $-1/16$.

9.2 Exercises

1. a. $\sqrt{279}$; $\pm\vec{u}/\sqrt{279}$; b. $\sqrt{341}$; $\pm\vec{u}/\sqrt{341}$; c. $\sqrt{131}$; $\pm\vec{u}/\sqrt{131}$; d. $\sqrt{354}$; $\pm p(x)/\sqrt{354}$; e. $\sqrt{1802}$; $\pm p(x)/\sqrt{1802}$
2. a. $\theta = \cos^{-1}\left(312/\sqrt{8051}\right)$ and $d(\vec{u},\vec{v}) = \sqrt{8}$; b. $\theta = \cos^{-1}\left(16/\sqrt{17510}\right)$ and $d(\vec{u},\vec{v}) = \sqrt{241}$;
   c. $\theta = \cos^{-1}(11/15)$ and $d(\vec{u},\vec{v}) = \sqrt{65}$; d. $\theta = \cos^{-1}\left(-298/\sqrt{131,334}\right)$ and $d(\vec{u},\vec{v}) = \sqrt{1321}$
   e. $\theta = \cos^{-1}\left(-10892/\sqrt{120816920}\right)$ and $d(\vec{u},\vec{v}) = \sqrt{47290}$
3. $8/\sqrt{15}$
4. $\cos(\theta) = 2/\sqrt{\pi^2 - 4}$, so $\theta \approx 0.6$ radians
5. $49x^2 + 25y^2 = 1$ is an ellipse (left, below):





6. $\dfrac{x^2}{49} + \dfrac{y^2}{25} = 1$ is an ellipse (above, right).
7. $4x^2 + y^2 + 25z^2 = 1$ is an ellipsoid with vertices $(\pm 1/2, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1/5)$
8. $\sqrt{7342}$
9. No. $19 > 15$, so the conditions violate the Cauchy-Schwarz Inequality.
10. $\|\vec{u}\| = 13$ and $\|\vec{v}\| = 5$.
14. c. You get an isosceles triangle.

9.3 Exercises

1. Answers:

   a. $\left\{\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{6}}\langle 2,-1,1\rangle, \frac{1}{\sqrt{2}}\langle 0,1,1\rangle\right\}$

   b. $\left\{\frac{1}{\sqrt{2}}\langle 1,0,1\rangle, \frac{1}{\sqrt{3}}\langle -1,1,1\rangle, \frac{1}{\sqrt{6}}\langle 1,2,-1\rangle\right\}$

   c. $\left\{\frac{1}{\sqrt{12}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{24}}\langle 2,-1,1\rangle, \frac{1}{\sqrt{120}}\langle 0,3,5\rangle\right\}$; different answer.

   d. $\left\{\frac{1}{\sqrt{7}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{581}}\langle 6,-7,-8\rangle, \frac{1}{\sqrt{4980}}\langle 15,24,-20\rangle\right\}$; different answer.

   e. $\left\{\frac{1}{\sqrt{5}}\langle 1,1,-1\rangle, \frac{1}{\sqrt{5}}\langle 2,-3,3\rangle, \langle 1,-1,2\rangle\right\}$; different answer.

   f. $\left\{\frac{1}{\sqrt{2}}\langle 1,0,1\rangle, \frac{1}{\sqrt{11}}\langle 2,-1,0\rangle, \frac{1}{\sqrt{22}}\langle -5,8,-11\rangle\right\}$

   g. $\left\{\frac{1}{2}\langle 1,-1,1,-1\rangle, \frac{1}{2\sqrt{11}}\langle 5,-1,-3,3\rangle, \frac{1}{\sqrt{330}}\langle 7,14,-2,-9\rangle, \frac{1}{\sqrt{30}}\langle 1,2,4,3\rangle\right\}$

   h. $\left\{\frac{1}{\sqrt{3}}\langle 1,-1,0,1\rangle, \frac{1}{\sqrt{15}}\langle 1,2,-3,1\rangle, \frac{1}{3\sqrt{10}}\langle 7,4,4,-3\rangle, \frac{1}{3\sqrt{2}}\langle -1,2,2,3\rangle\right\}$

   i. $\left\{\frac{1}{\sqrt{14}}\langle 1,-1,1,-1\rangle, \frac{1}{\sqrt{2198}}\langle 19,-5,-9,9\rangle,\right.$
   $\left.\frac{1}{\sqrt{125286}}\langle 33,264,-90,-67\rangle, \frac{1}{2\sqrt{399}}\langle 3,24,16,6\rangle\right\}$

   j. $\left\{\frac{1}{\sqrt{11}}\langle 1,-1,0,1\rangle, \frac{1}{\sqrt{473}}\langle 1,10,-11,1\rangle,\right.$
   $\left.\frac{1}{\sqrt{21930}}\langle 57,54,18,-29\rangle, \frac{1}{\sqrt{1020}}\langle -3,24,8,6\rangle\right\}$; different answer.

   k. $\left\{\frac{1}{\sqrt{17}}x^2, \frac{1}{6\sqrt{17}}(7x^2+17x), \frac{1}{2}(x^2+x-2)\right\}$

   l. $\left\{\frac{1}{\sqrt{30}}(x^2+1), \frac{1}{\sqrt{330}}(8x^2+15x-7), \frac{1}{\sqrt{99}}(4x^2+2x-9)\right\}$

   m. $\left\{\sqrt{5}\,x^2, \sqrt{3}\,(5x^2-4x), 10x^2-12x+3\right\}$

   n. $\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{42}}(3x+1), \frac{1}{\sqrt{126}}(7x^2+8x-9)\right\}$; different answer.

   o. $\left\{\sqrt{7}\,x^3, \sqrt{5}\,(6x^2-7x^3), \sqrt{3}\,(21x^3-30x^2+10x), -35x^3+60x^2-30x+4\right\}$

2. Answers:

   a. $\langle\vec{u}\rangle_S = \left\langle -\sqrt{3}, 3\sqrt{6}/2, -3\sqrt{2}/2\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle -2\sqrt{3}, -\sqrt{6}/2, 13\sqrt{2}/2\right\rangle$

   b. $\langle\vec{u}\rangle_S = \left\langle \frac{3}{\sqrt{2}}, -\frac{5}{\sqrt{3}}, -\frac{7}{\sqrt{6}}\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{5}{\sqrt{2}}, \frac{16}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right\rangle$

   c. $\langle\vec{u}\rangle_S = \left\langle -\frac{15}{\sqrt{12}}, \frac{39}{\sqrt{24}}, -\frac{45}{\sqrt{120}}\right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle -\frac{11}{\sqrt{12}}, -\frac{25}{\sqrt{24}}, \frac{195}{\sqrt{120}}\right\rangle$

d. $\langle\vec{u}\rangle_S = \left\langle 11\sqrt{7}, \frac{21}{\sqrt{581}}, -\frac{1530}{\sqrt{1245}} \right\rangle$, and

$\langle\vec{v}\rangle_S = \left\langle 12\sqrt{7}, -\frac{595}{\sqrt{581}}, -\frac{1470}{\sqrt{1245}} \right\rangle$

e. $\langle\vec{u}\rangle_S = \left\langle \frac{1}{\sqrt{5}}, \frac{12}{\sqrt{5}}, -3 \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle -\frac{12}{\sqrt{5}}, -\frac{34}{\sqrt{5}}, 13 \right\rangle$

f. $\langle\vec{u}\rangle_S = \left\langle -\frac{5}{\sqrt{2}}, \frac{16}{\sqrt{11}}, -\frac{7}{\sqrt{22}} \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{15}{\sqrt{2}}, -\frac{59}{\sqrt{11}}, -\frac{1}{\sqrt{22}} \right\rangle$

g. $\langle\vec{u}\rangle_S = \left\langle -\frac{1}{2}, \frac{3}{2\sqrt{11}}, \frac{145}{\sqrt{330}}, \frac{-5}{\sqrt{30}} \right\rangle$, and

$\langle\vec{v}\rangle_S = \left\langle \frac{17}{2}, \frac{-3}{2\sqrt{11}}, \frac{20}{\sqrt{330}}, \frac{20}{\sqrt{30}} \right\rangle$

h. $\langle\vec{u}\rangle_S = \left\langle -\frac{7}{3}\sqrt{3}, \frac{17}{15}\sqrt{15}, \frac{49}{30}\sqrt{10}, -\frac{7}{6}\sqrt{2} \right\rangle$, and

$\langle\vec{v}\rangle_S = \left\langle \frac{4}{3}\sqrt{3}, -\frac{23}{15}\sqrt{15}, \frac{32}{15}\sqrt{10}, -\frac{2}{3}\sqrt{2} \right\rangle$

i. $\langle\vec{u}\rangle_S = \left\langle \frac{24}{\sqrt{14}}, \frac{35}{\sqrt{2198}}, \frac{4128}{\sqrt{125286}}, -\frac{15}{\sqrt{399}} \right\rangle$, and

$\langle\vec{v}\rangle_S = \left\langle \frac{61}{\sqrt{14}}, \frac{39}{\sqrt{2198}}, -\frac{552}{\sqrt{125286}}, \frac{120}{\sqrt{399}} \right\rangle$

j. $\langle\vec{u}\rangle_S = \left\langle \frac{-18}{\sqrt{11}}, \frac{114}{\sqrt{473}}, \frac{1596}{\sqrt{21930}}, -\frac{84}{\sqrt{1020}} \right\rangle$,

and $\langle\vec{v}\rangle_S = \left\langle \frac{4}{\sqrt{11}}, -\frac{249}{\sqrt{473}}, \frac{1932}{\sqrt{21930}}, -\frac{48}{\sqrt{1020}} \right\rangle$

k. $\langle\vec{u}\rangle_S = \left\langle -\frac{56}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, 5 \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{73}{\sqrt{17}}, \frac{-48}{\sqrt{17}}, 4 \right\rangle$

l. $\langle\vec{u}\rangle_S = \left\langle \frac{-78}{\sqrt{30}}, \frac{36}{\sqrt{330}}, \frac{6}{\sqrt{99}} \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{82}{\sqrt{30}}, \frac{-184}{\sqrt{330}}, \frac{117}{\sqrt{99}} \right\rangle$

m. $\langle\vec{u}\rangle_S = \left\langle -\frac{41}{30}\sqrt{5}, \frac{3}{2}\sqrt{3}, -\frac{5}{3} \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle -\frac{67}{30}\sqrt{5}, \frac{11}{6}\sqrt{3}, -\frac{4}{3} \right\rangle$

n. $\langle\vec{u}\rangle_S = \left\langle \frac{-22}{\sqrt{3}}, \frac{44}{\sqrt{42}}, \frac{-18}{\sqrt{126}} \right\rangle$, and $\langle\vec{v}\rangle_S = \left\langle \frac{9}{\sqrt{3}}, \frac{-132}{\sqrt{42}}, \frac{54}{\sqrt{126}} \right\rangle$

o. $\langle\vec{u}\rangle_S = \left\langle \frac{461}{294}\sqrt{7}, \frac{52}{105}\sqrt{5}, \frac{29}{210}\sqrt{3}, -\frac{3}{35} \right\rangle$, and

$\langle\vec{v}\rangle_S = \left\langle \frac{433}{1470}\sqrt{7}, -\frac{1}{21}\sqrt{5}, -\frac{11}{210}\sqrt{3}, \frac{1}{7} \right\rangle$

3. a. $\int_0^{2\pi} \sin(x)\cos(x)dx = \int_0^{2\pi} \sin(2x)\cos(x)dx = \int_0^{2\pi} \sin(2x)\sin(x)dx = 0$,

$\int_0^{2\pi} \sin^2(2x)dx = \int_0^{2\pi} \sin^2(x)dx = \int_0^{2\pi} \cos^2(x)dx = \pi$

b. $\left\{ \frac{1}{\sqrt{\pi}}\sin(x), \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\sin(2x) \right\}$

c. $\langle\vec{u}\rangle_S = \sqrt{\pi}\langle 2, 7, -3\rangle$, and $\langle\vec{v}\rangle_S = \sqrt{\pi}\langle 5, -2, 1\rangle$

4. B is linearly dependent, because $\vec{w}_3$ is in the Span of $\{\vec{w}_1, \vec{w}_2\}$.

6. $u_1(x) = x(x-1)(x-2)/(-24)$, $u_2(x) = (x+2)(x-1)(x-2)/4$, $u_3(x) = (x+2)x(x-2)/(-3)$, $u_4(x) = (x+2)x(x-1)/8$.

9.4 Exercises

1. Answers:
   a. $\{\langle 1,-1,1\rangle\}$
   b. $\{\langle 1,1,0\rangle,\langle -3,0,1\rangle\}$
   c. $\{\langle 15,-12,20\rangle\}$
   d. $\{\langle 5,4,0\rangle,\langle -3,0,2\rangle\}$
   e. $\{\langle 1,0,1,0\rangle,\langle -1,-2,0,1\rangle\}$
   f. $\{\langle 3,0,4,0\rangle,\langle -3,-24,0,2\rangle\}$
   g. $\{\langle -9,-24,-8,2\rangle\}$
   h. $\{\langle 2,3,-3\rangle\}$
   i. $\{\langle 3,5,0\rangle,\langle 1,0,1\rangle\}$
   j. $\{7x^2+17x,-5x^2+17\}$
   k. $\left\{5x^2+7x-9\right\}$
   l. $\{224+888x-251x^2,414-3827x+251x^3\}$
   m. $\{1-5x,8-5x^2,7-5x^3\}$
   n. $\{25x^2-17x,50x^2-17\}$
2. Answers:
   a. for $W:\left\{\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle,\frac{1}{\sqrt{6}}\langle 2,-1,1\rangle\right\}$; for $W^\perp:\left\{\frac{1}{\sqrt{2}}\langle 0,1,1\rangle\right\}$
   b. for $W:\left\{\frac{1}{\sqrt{2}}\langle 1,0,1\rangle\right\}$; for $W^\perp:\left\{,\frac{1}{\sqrt{3}}\langle -1,1,1\rangle,\frac{1}{\sqrt{6}}\langle 1,2,-1\rangle\right\}$
   c. for $W:\left\{\frac{1}{2\sqrt{3}}\langle 1,1,-1\rangle\right\}$; for $W^\perp:\left\{\frac{1}{\sqrt{24}}\langle 2,-1,1\rangle,\frac{1}{\sqrt{120}}\langle 0,3,5\rangle\right\}$
   d. for $W:\left\{\frac{1}{\sqrt{7}}\langle 1,1,-1\rangle,\frac{1}{\sqrt{581}}\langle 6,-7,-8\rangle\right\}$; for $W^\perp:\left\{\frac{1}{\sqrt{4980}}\langle 15,24,-20\rangle\right\}$
   e. for $W:\left\{\frac{1}{\sqrt{2}}\langle 1,0,1\rangle,\frac{1}{\sqrt{11}}\langle 2,-1,0\rangle\right\}$; for $W^\perp:\left\{\frac{1}{\sqrt{22}}\langle -5,8,-11\rangle\right\}$
   f. for $W:\left\{\frac{1}{2}\langle 1,-1,1,-1\rangle,\frac{1}{2\sqrt{11}}\langle 5,-1,-3,3\rangle\right\}$;
      for $W^\perp:\left\{\frac{1}{\sqrt{330}}\langle 7,14,-2,-9\rangle,\frac{1}{\sqrt{30}}\langle 1,2,4,3\rangle\right\}$
   g. for $W:\left\{\frac{1}{\sqrt{3}}\langle 1,-1,0,1\rangle\right\}$;
      for $W^\perp:\left\{\frac{1}{\sqrt{15}}\langle 1,2,-3,1\rangle,\frac{1}{3\sqrt{10}}\langle 7,4,4,-3\rangle,\frac{1}{3\sqrt{2}}\langle -1,2,2,3\rangle\right\}$
   h. for
      $W:\left\{\frac{1}{\sqrt{14}}\langle 1,-1,1,-1\rangle,\frac{1}{\sqrt{2198}}\langle 19,-5,-9,9\rangle,\frac{1}{\sqrt{125286}}\langle 33,264,-90,-67\rangle\right\}$;
      for $W^\perp:\left\{\frac{1}{2\sqrt{399}}\langle 3,24,16,6\rangle\right\}$

i. for W : $\left\{\frac{1}{\sqrt{17}}x^2\right\}$; for $W^\perp$: $\left\{\frac{1}{6\sqrt{17}}(7x^2+17x), \frac{1}{2}(x^2+x-2)\right\}$

j. for W : $\left\{\frac{1}{\sqrt{30}}(x^2+1), \frac{1}{\sqrt{330}}(8x^2+15x-7)\right\}$;

for $W^\perp$: $\left\{\frac{1}{\sqrt{99}}(4x^2+2x-9)\right\}$

k. for W : $\left\{\sqrt{5}\,x^2, \sqrt{3}\,(5x^2-4x)\right\}$; for $W^\perp$: $\{10x^2-12x+3\}$

3. Answers:

a. Start with $\{\langle 5,-2,0\rangle, \vec{e}_1, \vec{e}_3\}$; for V : $\left\{\frac{1}{\sqrt{29}}\langle 5,-2,0\rangle, \frac{1}{\sqrt{29}}\langle 2,5,0\rangle, \vec{e}_3\right\}$;

for W : $\left\{\frac{1}{\sqrt{29}}\langle 5,-2,0\rangle\right\}$; for $W^\perp$: $\left\{\frac{1}{\sqrt{29}}\langle 2,5,0\rangle, \vec{e}_3\right\}$

b. Start with $\{\langle 5,-2,0\rangle, \vec{e}_1, \vec{e}_3\}$; for V : $\left\{\langle 5,-2,0\rangle/\sqrt{120}, \langle 1,2,0\rangle/\sqrt{24}, \langle 0,0,1\rangle/\sqrt{2}\right\}$;

for W : $\left\{\langle 5,-2,0\rangle/\sqrt{120}\right\}$; for $W^\perp$: $\left\{\langle 1,2,0\rangle/\sqrt{24}, \langle 0,0,1\rangle/\sqrt{2}\right\}$

c. Start with $\{\langle 1,-1,0,1\rangle, \langle 1,0,-3,1\rangle, \vec{e}_1, \vec{e}_2\}$;

For V : $\left\{\langle 1,-1,0,1\rangle/\sqrt{3}, \langle 1,2,-9,1\rangle/\sqrt{87}, \langle 19,9,3,-10\rangle/\sqrt{551}, \langle 0,3,1,3\rangle/\sqrt{19}\right\}$

for W : $\left\{\langle 1,-1,0,1\rangle/\sqrt{3}, \langle 1,2,-9,1\rangle/\sqrt{87}\right\}$;

for $W^\perp$: $\left\{\langle 19,9,3,-10\rangle/\sqrt{551}, \langle 0,3,1,3\rangle/\sqrt{19}\right\}$

d. Start with $\{\langle 1,-1,0,1\rangle, \langle 1,0,-3,1\rangle, \vec{e}_1, \vec{e}_2\}$;

For V : $\left\{\frac{1}{\sqrt{11}}\langle 1,-1,0,1\rangle, \frac{1}{\sqrt{3377}}\langle 1,10,-33,1\rangle, \frac{1}{2\sqrt{59865}}\langle 195,108,12,-112\rangle,\right.$

$\left.\frac{1}{\sqrt{104\,409\,309\,290}}\langle -97\,825, 115\,898, 57\,222, 84\,533\rangle\right\}$

for W : $\left\{\frac{1}{\sqrt{11}}\langle 1,-1,0,1\rangle, \frac{1}{\sqrt{3377}}\langle 1,10,-33,1\rangle\right\}$;

for $W^\perp$: $\left\{\frac{1}{2\sqrt{59865}}\langle 195,108,12,-112\rangle,\right.$

$\left.\frac{1}{\sqrt{104\,409\,309\,290}}\langle -97\,825, 115\,898, 57\,222, 84\,533\rangle\right\}$

e. Start with $\{x^2+5x, x^2, 1\}$; for V : $\left\{\frac{1}{\sqrt{72}}(x^2+5x), \frac{1}{\sqrt{8}}(x^2+x), \frac{1}{2}(x^2+x-2)\right\}$;

for W : $\left\{\frac{1}{\sqrt{72}}(x^2+5x)\right\}$; for $W^\perp$: $\left\{\frac{1}{\sqrt{8}}(x^2+x), \frac{1}{2}(x^2+x-2)\right\}$

f. Start with $\{x^2-3x, x, 1\}$;

for V : $\left\{\sqrt{\tfrac{10}{17}}\,(x^2-3x), \sqrt{\tfrac{6}{17}}\,(15x^2-11x), 10x^2-12x+3\right\}$;

for W : $\left\{\sqrt{\tfrac{10}{17}}\,(x^2-3x), \sqrt{\tfrac{6}{17}}\,(15x^2-11x)\right\}$; for $W^\perp$: $\{10x^2-12x+3\}$

4. Answers:

a. $\vec{w}_1 = \langle 2,-5/2,5/2\rangle$, and $\vec{w}_2 = \langle 0,-3/2,-3/2\rangle$

b. $\vec{w}_1 = \langle 5/2,0,5/2\rangle$, and $\vec{w}_2 = \langle -11/2,5,11/2\rangle$

c. $\vec{w}_1 = -\frac{5}{4}\langle 1,1,-1\rangle$, and $\vec{w}_2 = \frac{1}{4}\langle 13,-11,-1\rangle$

d. $\vec{w}_1 = \frac{1}{22}\langle -71, 118, 165\rangle$, and $\vec{w}_2 = -\frac{1}{22}\langle -5, 8, -11\rangle$
e. $\vec{w}_1 = \left\langle \frac{1}{11}, \frac{2}{11}, -\frac{5}{11}, \frac{5}{11} \right\rangle$, and $\vec{w}_2 = \left\langle \frac{32}{11}, \frac{64}{11}, -\frac{17}{11}, -\frac{49}{11} \right\rangle$
f. $\vec{w}_1 = \left\langle \frac{4}{3}, -\frac{4}{3}, 0, \frac{4}{3} \right\rangle$, and $\vec{w}_2 = \left\langle \frac{11}{3}, -\frac{2}{3}, 7, -\frac{13}{3} \right\rangle$
g. $\vec{w}_1 = \left\langle -\frac{15}{133}, -\frac{120}{133}, -\frac{80}{133}, -\frac{30}{133} \right\rangle$, and $\vec{w}_2 = \left\langle \frac{414}{133}, \frac{918}{133}, -\frac{186}{133}, -\frac{502}{133} \right\rangle$
h. $\vec{w}_1 = -56x^2/17$, and $\vec{w}_2 = 39x^2/17 + 2x - 5$
i. $\vec{w}_1 = \frac{49}{3}x^2 - 22x$, and $\vec{w}_2 = -\frac{40}{3}x^2 + 16x - 4$

9.5 Exercises

1. Answers:
   a. (i) $\langle \vec{u} \mid \vec{v} \rangle = -100$; (ii) $\|\vec{u}\| = 3\sqrt{11}$; (iii) $\|\vec{v}\| = \sqrt{353}$; (iv) $d(\vec{u},\vec{v}) = 2\sqrt{163}$; (v) $\cos(\theta) = -\frac{100}{3\sqrt{11}\sqrt{353}}$
   b. (i) $\langle \vec{u} \mid \vec{v} \rangle = -123$; (ii) $\|\vec{u}\| = \sqrt{38}$; (iii) $\|\vec{v}\| = \sqrt{429}$; (iv) $d(\vec{u},\vec{v}) = \sqrt{713}$; (v) $\cos(\theta) = -\frac{123}{\sqrt{38}\sqrt{429}}$
   c. (i) $\langle \vec{u} \mid \vec{v} \rangle = 78$; (ii) $\|\vec{u}\| = 6\sqrt{5}$; (iii) $\|\vec{v}\| = \sqrt{305}$; (iv) $d(\vec{u},\vec{v}) = \sqrt{329}$; (v) $\cos(\theta) = \frac{13}{5\sqrt{61}}$
   d. (i) $\langle \vec{u} \mid \vec{v} \rangle = -212$; (ii) $\|\vec{u}\| = \sqrt{210}$; (iii) $\|\vec{v}\| = \sqrt{465}$; (iv) $d(\vec{u},\vec{v}) = \sqrt{1099}$; (v) $\cos(\theta) = -\frac{212}{\sqrt{97650}}$
   e. (i) $\langle \vec{u} \mid \vec{v} \rangle = \frac{386}{15}$; (ii) $\|\vec{u}\| = \frac{1}{15}\sqrt{4245}$; (iii) $\|\vec{v}\| = \frac{2}{5}\sqrt{230}$; (iv) $d(\vec{u},\vec{v}) = \frac{1}{5}\sqrt{105}$; (v) $\cos(\theta) = \frac{193}{\sqrt{39054}}$
2. Answers:
   a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$
   b. $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$
   c. $\begin{bmatrix} \frac{9}{11} & -\frac{4}{11} & -\frac{1}{11} & \frac{1}{11} \\ -\frac{4}{11} & \frac{3}{11} & -\frac{2}{11} & \frac{2}{11} \\ -\frac{1}{11} & -\frac{2}{11} & \frac{5}{11} & -\frac{5}{11} \\ \frac{1}{11} & \frac{2}{11} & -\frac{5}{11} & \frac{5}{11} \end{bmatrix}$

d. $\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$

3. $\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}$

4. One way is to apply Gram-Schmidt to $\{\langle 3,5,0\rangle, \langle 7,0,5\rangle\}$; we get: $\begin{bmatrix} \frac{58}{83} & \frac{15}{83} & \frac{35}{83} \\ \frac{15}{83} & \frac{74}{83} & -\frac{21}{83} \\ \frac{35}{83} & -\frac{21}{83} & \frac{34}{83} \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{16}{65} & \frac{28}{65} \\ 0 & \frac{28}{65} & \frac{49}{65} \end{bmatrix}$

14. c. $f(x) = -7x^4 + 5x^2 - 1$, and $g(x) = 8x^5 - 2x^3 + 6x$

9.6 Exercises

1. $\begin{bmatrix} -8/17 & 15/17 \\ 15/17 & 8/17 \end{bmatrix}$ is improper, while $\begin{bmatrix} -8/17 & -15/17 \\ 15/17 & -8/17 \end{bmatrix}$ is proper.

2. $\begin{bmatrix} 20/29 & -21/29 \\ -21/29 & -20/29 \end{bmatrix}$ is improper, while $\begin{bmatrix} 20/29 & -21/29 \\ 21/29 & 20/29 \end{bmatrix}$ is proper.

3. $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (improper).

4. $$\begin{bmatrix} \frac{1}{2} & \frac{5}{2\sqrt{11}} & \frac{7}{\sqrt{330}} & \frac{1}{\sqrt{30}} \\ -\frac{1}{2} & -\frac{1}{2\sqrt{11}} & \frac{14}{\sqrt{330}} & \frac{2}{\sqrt{30}} \\ \frac{1}{2} & -\frac{3}{2\sqrt{11}} & \frac{-2}{\sqrt{330}} & \frac{4}{\sqrt{30}} \\ -\frac{1}{2} & \frac{3}{2\sqrt{11}} & \frac{-9}{\sqrt{330}} & \frac{3}{\sqrt{30}} \end{bmatrix} \text{(proper)}$$

5. b. $Q = \begin{bmatrix} -20/29 & 21/29 \\ 21/29 & 20/29 \end{bmatrix}$ and $Q' = \begin{bmatrix} 15/17 & -8/17 \\ 8/17 & 15/17 \end{bmatrix}$

c. Q is improper and Q' is proper.

d. $QQ' = \begin{bmatrix} -\frac{132}{493} & \frac{475}{493} \\ \frac{475}{493} & \frac{132}{493} \end{bmatrix}$. e. QQ' is improper. f. $C_{B,B'} = \begin{bmatrix} -\frac{132}{493} & \frac{475}{493} \\ \frac{475}{493} & \frac{132}{493} \end{bmatrix}$

g. $C_{B,B'}$ is improper.

6. $$C_{B,B'} = \begin{bmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ -\frac{5}{42}\sqrt{42} & \frac{1}{42}\sqrt{42} & \frac{2}{21}\sqrt{42} \\ \frac{1}{14}\sqrt{14} & -\frac{3}{14}\sqrt{14} & \frac{1}{7}\sqrt{14} \end{bmatrix}$$

7. There are 2^n possible combinations.
8. There are $n!$ such rearrangements.
18. Erratum: in part (c): change $\langle 4,-2,-1\rangle$ to $\langle 4,-2,1\rangle$

c. $\vec{w} = \langle 1,-11,-26\rangle/\sqrt{798}$

e. $Q = \begin{bmatrix} 3/\sqrt{38} & 4/\sqrt{21} & 1/\sqrt{798} \\ 5/\sqrt{38} & -2/\sqrt{21} & -11/\sqrt{798} \\ -2/\sqrt{38} & 1/\sqrt{21} & -26/\sqrt{798} \end{bmatrix}$

g. $\vec{v} = \langle -ac,-bc, a^2+b^2\rangle/\sqrt{a^2+b^2}$

h. $\left\{ \frac{1}{7}\langle 3,-2,6\rangle, \frac{1}{\sqrt{13}}\langle 2,3,0\rangle, \frac{1}{7\sqrt{13}}\langle -18,12,13\rangle \right\}$

9.7 Exercises

1. Answers:

a. $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$; $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

b. $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}; D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

c. $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}; D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

d. $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & -\frac{4}{\sqrt{18}} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix}; D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 11 \end{bmatrix}$

e. $Q = \begin{bmatrix} \frac{2}{\sqrt{17}} & -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{34}} \\ -\frac{3}{\sqrt{17}} & 0 & \frac{4}{\sqrt{34}} \\ \frac{2}{\sqrt{17}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{34}} \end{bmatrix}; D = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

f. $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{22}} \\ 0 & -\frac{3}{\sqrt{11}} & \frac{2}{\sqrt{22}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{22}} \end{bmatrix}; D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 16 \end{bmatrix}$

g. $Q = \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}; D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

h. $Q = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}; D = \begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

i. $Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}; D = \begin{bmatrix} -10 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

j. $Q = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}; D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

k. $Q = \begin{bmatrix} -\frac{3}{\sqrt{62}} & \frac{-27}{\sqrt{6510}} & \frac{2}{\sqrt{22}} & \frac{36}{\sqrt{2310}} \\ -\frac{2}{\sqrt{62}} & \frac{44}{\sqrt{6510}} & -\frac{3}{\sqrt{22}} & \frac{23}{\sqrt{2310}} \\ 0 & \frac{62}{\sqrt{6510}} & \frac{3}{\sqrt{22}} & -\frac{1}{\sqrt{2310}} \\ \frac{7}{\sqrt{62}} & \frac{1}{\sqrt{6510}} & 0 & \frac{22}{\sqrt{2310}} \end{bmatrix}; D = \begin{bmatrix} -42 & 0 & 0 & 0 \\ 0 & -42 & 0 & 0 \\ 0 & 0 & 63 & 0 \\ 0 & 0 & 0 & 63 \end{bmatrix}$

l. $Q = \begin{bmatrix} \frac{3}{\sqrt{11}} & -\frac{3}{\sqrt{231}} & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{210}} \\ -\frac{1}{\sqrt{11}} & \frac{1}{\sqrt{231}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{210}} \\ \frac{1}{\sqrt{11}} & \frac{10}{\sqrt{231}} & 0 & -\frac{10}{\sqrt{210}} \\ 0 & \frac{11}{\sqrt{231}} & 0 & \frac{10}{\sqrt{210}} \end{bmatrix}; D = \begin{bmatrix} -63 & 0 & 0 & 0 \\ 0 & -63 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 21 \end{bmatrix}$

m. $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ 0 & 0 & \frac{3}{\sqrt{12}} & \frac{1}{2} \end{bmatrix}; D = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

n. $Q = \begin{bmatrix} -\frac{5}{\sqrt{35}} & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{42}} & \frac{5}{\sqrt{210}} \\ \frac{1}{\sqrt{35}} & -\frac{1}{\sqrt{7}} & \frac{5}{\sqrt{42}} & \frac{7}{\sqrt{210}} \\ 0 & \frac{1}{\sqrt{7}} & \frac{4}{\sqrt{42}} & -\frac{10}{\sqrt{210}} \\ \frac{3}{\sqrt{35}} & \frac{2}{\sqrt{7}} & 0 & \frac{6}{\sqrt{210}} \end{bmatrix}; D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 35 & 0 \\ 0 & 0 & 0 & 35 \end{bmatrix}$

o. $Q = \begin{bmatrix} \frac{3}{\sqrt{58}} & -\frac{7}{\sqrt{899}} & \frac{-7}{3\sqrt{434}} & \frac{7}{3\sqrt{7}} \\ \frac{7}{\sqrt{58}} & \frac{3}{\sqrt{899}} & \frac{3}{3\sqrt{434}} & -\frac{3}{3\sqrt{7}} \\ 0 & \frac{29}{\sqrt{899}} & -\frac{2}{3\sqrt{434}} & \frac{2}{3\sqrt{7}} \\ 0 & 0 & \frac{62}{3\sqrt{434}} & \frac{1}{3\sqrt{7}} \end{bmatrix}; D = \begin{bmatrix} -63 & 0 & 0 & 0 \\ 0 & -63 & 0 & 0 \\ 0 & 0 & -63 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

p. $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{7}{\sqrt{123}} & -\frac{1}{2\sqrt{574}} & -\frac{2}{\sqrt{15}} & \frac{1}{2\sqrt{210}} \\ -\frac{1}{\sqrt{3}} & -\frac{8}{\sqrt{123}} & -\frac{7}{2\sqrt{574}} & \frac{1}{\sqrt{15}} & \frac{7}{2\sqrt{210}} \\ 0 & \frac{3}{\sqrt{123}} & -\frac{23}{2\sqrt{574}} & -\frac{1}{\sqrt{15}} & \frac{23}{2\sqrt{210}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{123}} & -\frac{6}{2\sqrt{574}} & \frac{3}{\sqrt{15}} & \frac{6}{2\sqrt{210}} \\ 0 & 0 & \frac{41}{2\sqrt{574}} & 0 & \frac{15}{2\sqrt{210}} \end{bmatrix};$

$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 56 & 0 \\ 0 & 0 & 0 & 0 & 56 \end{bmatrix}$

q. $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$; $D = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$

r. $Q = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$; $D = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$

s. $Q = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$; $D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$

There is exactly one different eigenvalue between the matrices in parts (r) and (s), but the diagonalizing orthogonal matrix is the same for both. This is explained further in Exercise 2 (f) and (g).

2. Erratum: to avoid the possibility of dividing by zero, a, b, c should not be zero in this problem. In parts (e) through (g), assume further that $\pm a$, $\pm b$, and c are all distinct.
   a. (i) $p(\lambda) = (\lambda - a - 2b)(\lambda - a + b)^2$
      (ii) eigenvalues $a + 2b$, with multiplicity 1, and $a - b$, with multiplicity 2.
      (iii) $\lambda = a + 2b : \{\langle 1, 1, 1\rangle\}$;
      $\lambda = a - b : \{\langle -1, 1, 0\rangle, \langle -1, 0, 1\rangle\}$
   b. (i) $p(\lambda) = (\lambda - a - 3b)(\lambda - a + b)^3$
      (ii) eigenvalues $a + 2b$, with multiplicity 1, and $a - b$, with multiplicity 3.
      (iii) $\lambda = a + 2b : \{\langle 1, 1, 1, 1\rangle\}$;
      $\lambda = a - b : \{\langle -1, 1, 0, 0\rangle, \langle -1, 0, 1, 0\rangle, \langle -1, 0, 0, 1\rangle\}$
   c. (i) $p(\lambda) = (\lambda - a)(\lambda + a)(\lambda - b)(\lambda + b)$
      (ii) eigenvalues $\pm a$, $\pm b$, all with multiplicity 1.

(iii) $\lambda = -a : \{\langle 1,0,0,1\rangle\}$; $\lambda = a : \{\langle -1,0,0,1\rangle\}$
$\lambda = -b : \{\langle 0,-1,1,0\rangle\}$; $\lambda = b : \{\langle 0,1,1,0\rangle\}$

d. (i) $p(\lambda) = (\lambda - a - b)^2(\lambda - a + b)^2$
(ii) eigenvalues $a + b$, $a - b$, both with multiplicity 2.
(iii) $\lambda = a + b : \{\langle 0,1,1,0\rangle, \langle 1,0,0,1\rangle\}$;
$\lambda = a - b : \{\langle 0,-1,1,0\rangle, \langle -1,0,0,1\rangle\}$

e. (i) $p(\lambda) = (\lambda - a)(\lambda + a)(\lambda - b)(\lambda + b)(\lambda - c)$
(ii) eigenvalues $\pm a$, $\pm b$, all with multiplicity 1.
(iii) $\lambda = -a : \{\langle -1,0,0,0,1\rangle\}$; $\lambda = a : \{\langle 1,0,0,0,1\rangle\}$;
$\lambda = -b : \{\langle 0,-1,0,1,0\rangle\}$; $\lambda = b : \{\langle 0,1,0,1,0\rangle\}$;
$\lambda = c : \{\langle 0,0,1,0,0\rangle\}$

f. (i) $p(\lambda) = (\lambda - a - b)^2(\lambda - a + b)^2(\lambda - a)$
(ii) eigenvalues $a + b$, $a - b$, both with multiplicity 2, and a with multiplicity 1.
(iii) $\lambda = a + b : \{\langle 0,1,0,1,0\rangle, \langle 1,0,0,0,1\rangle\}$;
$\lambda = a - b : \{\langle 0,-1,0,1,0\rangle, \langle -1,0,0,0,1\rangle\}$
$\lambda = a : \{\langle 0,0,1,0,0\rangle\}$

g. (i) $p(\lambda) = (\lambda - a - b)^2(\lambda - a + b)^2(\lambda - b)$
(ii) eigenvalues $a + b$, $a - b$, both with multiplicity 2, and b with multiplicity 1.
(iii) $\lambda = a + b : \{\langle 0,1,0,1,0\rangle, \langle 1,0,0,0,1\rangle\}$;
$\lambda = a - b : \{\langle 0,-1,0,1,0\rangle, \langle -1,0,0,0,1\rangle\}$
$\lambda = a : \{\langle 0,0,1,0,0\rangle\}$

3. Eigenvalues $\pm c_1, \pm c_2, \ldots \pm c_{k-1}, c_k$, all with multiplicity 1, where $k = n/2$.
4. Eigenvalues $\pm c_1, \pm c_2, \ldots \pm c_{k-1}, c_k$, all with multiplicity 1, where $k = (n+1)/2$.
5. Eigenvalues $a \pm b$, each with multiplicity $n/2$.
6. a. Eigenvalues a, with multiplicity 1, and $a \pm b$, each with multiplicity $(n-1)/2$.
b. Eigenvalues b, with multiplicity 1, and $a \pm b$, each with multiplicity $(n-1)/2$.

Chapter Ten Exercises

10.1 Exercises

1. Answers:
   a. $7 + 4i$
   b. $31 + 29i$
   c. $-\frac{2}{13} + \frac{29}{13}i$
   d. 25
   e. $-i$
   f. i
2. Answers:
   a. 4096
   b. $3^{17}(-256)(1 + i)$
   c. $-\frac{\sqrt{3}}{256} + \frac{1}{256}i$
   d. $164833 - 354144i$
3. Answers:
   a. $z = -2 \pm 2i\sqrt{3}$
   b. $z = \sqrt{2} + i\sqrt{2}$ and $-\frac{\sqrt{2}}{2} \pm \frac{\sqrt{6}}{2} + i\left(-\frac{\sqrt{2}}{2} \mp \frac{\sqrt{6}}{2}\right)$
   c. $z = \frac{3}{2}\sqrt{2} \pm \frac{3}{2}i\sqrt{2}$ and $-\frac{3}{2}\sqrt{2} \pm \frac{3}{2}i\sqrt{2}$
   d. $z = \pm(3 - 4i)$
4. Answers:
   a. $z = \frac{3}{2} - \frac{5}{2}i$ and $-2 + 3i$
   b. $z = \frac{5}{3} + \frac{2}{3}i$ and $-\frac{3}{2} + 2i$
   c. $z = -\frac{3}{5} + \frac{1}{5}i$ and $2 + i$

10.2 Exercises

1. Answers:
   a. Yes: $\vec{b} = i\langle 1 - i, 2i, 3\rangle - i\langle -i, 3i, 2\rangle$
   b. No.
   c. Yes. $\vec{b} = (1 + 2i)\langle 1 - i, i, 3, -2i\rangle + (1 - i)\langle 2i, -1, i, 3i\rangle + (2 + 3i)\langle 2, 3i, 6 + i, -i\rangle$
   d. Yes. $\vec{b} = (2 + 2i)[1 - i + 2iz - 3z^2] + (-2 + i)[2i + 3z + (i - 1)z^2]$
   e. Yes. $\vec{b} = (5 + 8i)[1 - i + 2iz - 3z^2] + (-4 + i)[2i + 3z + (i - 1)z^2]$
2. Answers:
   a. Dependent; $(2i)\langle 1 - i, 2i, 3\rangle = \langle 2 + 2i, -4, 6i\rangle$; $\dim(W) = 2$.
   b. Independent; $\dim(W) = 3$.
   c. Dependent; $\begin{bmatrix} -1 + i & -2 - 2i \\ 1 + i & 2 \end{bmatrix} = (1 + i)\begin{bmatrix} i & -2 \\ 1 & 1 - i \end{bmatrix}$; $\begin{bmatrix} -1 + 3i & -2 - 2i \\ 1 + i & 2i \end{bmatrix} = i\begin{bmatrix} i & -2 \\ 1 & 1 - i \end{bmatrix} + i\begin{bmatrix} 3 & 2i \\ -i & 1 + i \end{bmatrix}$; $\dim(W) = 2$

d. Dependent; $1 - i + 2iz - 3z^2 + i[2i + 3z + (i-1)z^2] = -1 - i + 5iz - (4+i)z^2$; $\dim(W) = 2$.

3. Answers:

a. Yes, W is a subspace; $\dim(W) = 1$; $\{-2 + 2i - (1+3i)z + z^2\}$

b. Not a subspace.

c. Yes, W is a subspace; $\dim(W) = 1$; $\{5 - 2z + z^2\}$

d. Yes, W is a subspace; $\dim(W) = 1$; $\{-1 - 4i + (2i-4)z + z^2\}$

e. Yes, W is a subspace; $\dim(W) = 1$; $\{5 - 2z + z^2\}$

f. Yes, W is a subspace; $\dim(W) = n^2 - 1$; to create a basis, note that the trace only cares about the diagonal entries. Thus, the $n^2 - n$ non-diagonal entries are free, and if $i \neq j$, the $n \times n$ matrix consisting of all zeroes except for a lone entry of 1 in row i column j will be a member of the basis. Now, we can make $a_{11}, a_{22}, \ldots, a_{n-1,n-1}$ to be all free, but we must have $a_{n,n} = -a_{11} - a_{22} - \cdots - a_{n-1,n-1}$. Thus, for $i = 1..n-1$, we will have another basis member of the form $Diag(0,\ldots,0,1,0,\ldots,0,-1)$ where 1 is in $a_{i,i}$. The total number of basis members is thus $n^2 - n + (n-1) = n^2 - 1 = dim(W)$.

4. Answers:

a. $A^2 = \begin{bmatrix} -2 & 4+2i \\ 3-i & -3+2i \end{bmatrix}$; $det(A) = 1 - 3i$; $A^{-1} = \begin{bmatrix} \frac{3}{10} - \frac{1}{10}i & \frac{3}{5} - \frac{1}{5}i \\ \frac{1}{5} - \frac{2}{5}i & \frac{2}{5} + \frac{1}{5}i \end{bmatrix}$

b. $A^2 = \begin{bmatrix} 4+5i & 0 \\ 0 & 4+5i \end{bmatrix}$; $det(A) = -4 - 5i$; $A^{-1} = \begin{bmatrix} -\frac{5}{41} - \frac{4}{41}i & \frac{3}{41} - \frac{14}{41}i \\ \frac{19}{41} + \frac{7}{41}i & \frac{5}{41} + \frac{4}{41}i \end{bmatrix}$

c. $A^2 = \begin{bmatrix} -3i & 6-3i \\ 3-3i & 9+3i \end{bmatrix}$; $det(A) = 0$, so A^{-1} does not exist.

d. $A^2 = \begin{bmatrix} 1 & -2i & 2-2i \\ 0 & 3-2i & 1-i \\ 2+2i & 0 & 1-2i \end{bmatrix}$;

$det(A) = -5i$; $A^{-1} = \begin{bmatrix} -\frac{2}{5} - \frac{1}{5}i & \frac{3}{5} - \frac{1}{5}i & \frac{2}{5} + \frac{1}{5}i \\ \frac{1}{5} + \frac{1}{5}i & -\frac{1}{5}i & \frac{2}{5} \\ -\frac{2}{5} + \frac{2}{5}i & \frac{2}{5} & -\frac{1}{5}i \end{bmatrix}$

e. $A^2 = \begin{bmatrix} -1+2i & -3+3i & 1+i \\ 2 & -6+5i & 1-i \\ -7-i & 11 & -5+3i \end{bmatrix}$;

$det(A) = -5 - 5i$; $A^{-1} = \begin{bmatrix} \frac{1}{5} + \frac{2}{5}i & -i & -\frac{1}{5} - \frac{2}{5}i \\ \frac{1}{5} - \frac{1}{5}i & 0 & -\frac{1}{5}i \\ \frac{2}{5} - \frac{3}{5}i & -\frac{1}{2} + \frac{1}{2}i & -\frac{1}{10} + \frac{1}{2}i \end{bmatrix}$

10.3 Exercises

1. b. $195 + 63i$; c. 292
2. b. $703 + 166i$; c. 1149
3. Answers:
   a. $\left\{ \frac{1}{\sqrt{6}}\langle i, 2 - i\rangle, \frac{1}{5\sqrt{6}}\langle 11 - 2i, 3 + 4i\rangle \right\}$
   b. $\left\{ \frac{1}{4}\langle 1 + i, 2, 3 - i\rangle, \frac{1}{2\sqrt{102}}\langle 1 - 14i, 11 + 9i, -3\rangle, \frac{1}{\sqrt{22695}}\langle 50 + 80i, 76 - 3i, -87 + 21i\rangle \right\}$
   c. $\left\{ \frac{1}{\sqrt{7}}\langle i, 2 + i, -i\rangle, \frac{1}{6\sqrt{14}}\langle 5 + i, 7 - 2i, 16 + 13i\rangle, \frac{1}{6\sqrt{34}}\langle 25 + 19i, -14 + 5i, 1 - 4i\rangle \right\}$
4. $\left\{ \frac{1}{\sqrt{181}}\langle i, 2 - i\rangle, \frac{1}{5\sqrt{32218}}\langle 81 + 717i, 408 - 344i\rangle \right\}$
5. $\left\{ \frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{21}}(3z - 4 - i), \frac{1}{30\sqrt{182}}(84z^2 - (207 + 93i)z + 105 + 249i) \right\}$
6. Answers:
   a. $\left\{ \frac{1}{5\sqrt{6}}\langle 11 - 2i, 3 + 4i\rangle \right\}$
   b. $\left\{ \frac{1}{\sqrt{22695}}\langle 50 + 80i, 76 - 3i, -87 + 21i\rangle \right\}$
   c. $\left\{ \frac{1}{6\sqrt{14}}\langle 5 + i, 7 - 2i, 16 + 13i\rangle, \frac{1}{6\sqrt{34}}\langle 25 + 19i, -14 + 5i, 1 - 4i\rangle \right\}$
   d. $\left\{ \frac{1}{5\sqrt{32218}}\langle 81 + 717i, 408 - 344i\rangle \right\}$
   e. $\left\{ \frac{1}{30\sqrt{182}}(84z^2 - (207 + 93i)z + 105 + 249i) \right\}$

10.4 Exercises

1. Answers:
   a. $ker(T) = Span(\{\langle -2 + 3i, 1\rangle\})$; $range(T) = Span(\{\langle 3 - i, 5 + 2i\rangle\})$; T is not one-to-one; T is not onto; T is not an isomorphism.
   b. $ker(T) = \left\{\vec{0}_2\right\}$ $range(T) = \mathbb{C}^2$; T is one-to-one; T is onto; T is an isomorphism; $[T^{-1}] = \begin{bmatrix} \frac{4}{7} + \frac{1}{7}i & -\frac{3}{7} \\ -\frac{2}{7}i & \frac{2}{7} + \frac{1}{7}i \end{bmatrix}$
   c. $ker(T) = \left\{\vec{0}_2\right\}$; $range(T) = Span(\{\langle -2i, 2 + i, 1 - 3i\rangle, \langle 2 - 6i, 5i, 9\rangle\})$; T is one-to-one; T is not onto; T is not an operator, so it cannot be an isomorphism.
   d. $ker(T) = Span(\{\langle -2 + 3i, 1, 0\rangle, \langle 2i, 3\rangle\})$; $range(T) = Span(\{\langle 3 - i, 4 + 3i\rangle, \})$; T is not one-to-one; T is onto; T is not an operator, so it cannot be an isomorphism.
   e. $ker(T) = Span(\{\langle -2i, 1\rangle\})$; $range(T) = Span(\{\langle 2 + i, 3, 3 - 5i\rangle\})$; T is not one-to-one; T is not onto; T is not an operator, so it cannot be an isomorphism.
   f. $ker(T) = Span(\{\langle -4i, -5, 1\rangle\})$; $range(T) = \mathbb{C}^2$; T is not one-to-one; T is onto; T is not an operator, so it cannot be an isomorphism.
   g. $ker(T) = Span(\{\langle 2i - 3, 1, 0\rangle\})$; $range(T) = Span(\{\langle 3 + 2i, 2 - i, 3i\rangle, \langle 5, 3i, -2\rangle\})$; T is not one-to-one; T is not onto; T is not an isomorphism.
   h. $ker(T) = Span(\{\langle -2i, i - 3, 1\rangle\})$; $range(T) = Span(\{\langle 1 - i, 3 + i, 3i\rangle, \langle 2i, 2 - i, 5\rangle\})$; T is not one-to-one; T is not onto; T is not an isomorphism.
   i. $ker(T) = \left\{\vec{0}_3\right\}$; $range(T) = \mathbb{C}^3$; T is one-to-one; T is onto; T is an isomorphism,

and $[T^{-1}] = \frac{1}{74}\begin{bmatrix} 29-11i & 5-7i & 14+10i \\ 17-9i & 31+i & -2-12i \\ -23+47i & 19+3i & -6-36i \end{bmatrix}$

2. Answers:
   a. $p(\lambda) = \lambda^2 - (8+i)\lambda + 21 - i$; $Spec(T) = \{5+3i, 3-2i\}$;
   $Eig(A, 5+3i) = Span(\{\langle 3+4i, 3\rangle\})$; $Eig(A, 3-2i) = Span(\{\langle 2+3i, 2\rangle\})$
   A is diagonalizable with $D = Diag(5+3i, 3-2i)$ and $C = \begin{bmatrix} 3+4i & 2+3i \\ 3 & 2 \end{bmatrix}$
   b. $p(\lambda) = \lambda^2 - 4i\lambda - 4$; $Spec(T) = \{2i\}$
   $Eig(A, 2i) = Span(\{\langle 6+3i, 5\rangle\})$; A is not diagonalizable.
3. $p(\lambda) = \lambda^2 - 2\cos(\theta)\lambda + 1$; $\lambda = \cos(\theta) \pm i \cdot \sin(\theta)$
   For $\lambda = \cos(\theta) - i \cdot \sin(\theta)$: $\{\langle i, 1\rangle\}$
   For $\lambda = \cos(\theta) + i \cdot \sin(\theta)$: $\{\langle -i, 1\rangle\}$
   $D = \begin{bmatrix} \cos(\theta) - i\cdot\sin(\theta) & 0 \\ 0 & \cos(\theta) + i\cdot\sin(\theta) \end{bmatrix}$; $C = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$

10.5 Exercises

1. Answers:
   a. Hermitian, hence normal; $det(A) = 6$ (pure real); $p(\lambda) = \lambda^2 - 7\lambda + 6$;
   $Spec(A) = \{1, 6\}$, all eigenvalues are pure real.
   b. Skew-Hermitian, hence normal; $det(A) = -6$ (pure real, since n is even);
   $p(\lambda) = \lambda^2 - 7i\lambda - 6$; $Spec(A) = \{i, 6i\}$, all eigenvalues are pure imaginary.
   c. symmetric, hence Hermitian, hence normal; $det(A) = 24$ (pure real);
   $p(\lambda) = \lambda^2 - 11\lambda + 24$; $Spec(A) = \{3, 8\}$, all eigenvalues are pure real.
   d. Not normal.
   e. Hermitian, hence normal; $det(A) = -42$ (pure real); $p(\lambda) = \lambda^2 - \lambda - 42$;
   $Spec(A) = \{-6, 7\}$, all eigenvalues are pure real.
   f. unitary, hence normal; $det(A) = 1$ (for unitary); $p(\lambda) = \lambda^2 - \frac{4}{3}\lambda + 1$;
   $Spec(A) = \left\{\frac{2}{3} + \frac{1}{3}i\sqrt{5}, \frac{2}{3} - \frac{1}{3}i\sqrt{5}\right\}$, complex numbers of length 1.
   g. normal; $det(A) = 13$; $p(\lambda) = \lambda^2 - 6\lambda + 13$; $Spec(A) = \{3+2i, 3-2i\}$
   h. Skew-Hermitian, hence normal; $det(A) = 1050$ (pure real, since n is even);
   $p(\lambda) = \lambda^2 - 5i\lambda + 1050$; $Spec(A) = \{-30i, 35i\}$, all eigenvalues are pure imaginary.
   i. unitary, hence normal; $det(A) = -i$ (for unitary); $p(\lambda) = \lambda^2 + \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\lambda - i$;
   $Spec(A) = \left\{-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} + \left(-\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)i, \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} + \left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)i\right\}$, both of length 1.
   j. Hermitian, hence normal; $det(A) = -4$ (pure real); $p(\lambda) = \lambda^3 - \lambda^2 - 4\lambda + 4$;
   $Spec(A) = \{1, 2, -2\}$, all eigenvalues are pure real.
   k. Skew-Hermitian, hence normal; $det(A) = 5i$ (pure imaginary, since n is odd);
   $p(\lambda) = \lambda^3 - i\lambda^2 + 5\lambda - 5i$; $Spec(A) = \left\{i\sqrt{5}, -i\sqrt{5}, i\right\}$, all eigenvalues are pure imaginary.

l. Hermitian, hence normal; $det(A) = 5$ (pure real); $p(\lambda) = \lambda^3 + \lambda^2 - 5\lambda - 5$;
$Spec(A) = \{-1, \sqrt{5}, -\sqrt{5}\}$, all eigenvalues are pure real.

m. unitary, hence normal; $det(A) = -1$ (for unitary);
$p(\lambda) = \lambda^3 + \left(-\frac{\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i\right)\lambda^2 + \left(-\frac{\sqrt{2}}{2} - \frac{2+\sqrt{2}}{2}i\right)\lambda + 1$
$= (\lambda + i)\left(\lambda^2 + \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\lambda - i\right)$;
$Spec(A) = \left\{-\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} + \left(-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right)i, \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} + \left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right)i, -i\right\}$, all of length 1.

n. normal; $det(A) = 170$; $p(\lambda) = \lambda^3 - 15\lambda^2 + 84\lambda - 170$;
$Spec(A) = \{5, 5+3i, 5-3i\}$

o. normal; $det(A) = 50$; $p(\lambda) = \lambda^3 - 3\lambda^2 + 52\lambda - 50$; $Spec(A) = \{1, 1+7i, 1-7i\}$

10.6 Exercises

1. Answers:

a. $D = Diag(1,6)$, $C = \begin{bmatrix} -\frac{i}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{i}{\sqrt{5}} \end{bmatrix}$

b. $D = Diag(i,6i)$, $C = \begin{bmatrix} \frac{i}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{i}{\sqrt{5}} \end{bmatrix}$

c. $D = Diag(3,8)$, $C = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

d. Not possible.

e. $D = Diag(-6,7)$, $C = \begin{bmatrix} \frac{-2i}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{-2i}{\sqrt{13}} \end{bmatrix}$

f. $D = Diag\left(\frac{2}{3} + \frac{1}{3}i\sqrt{5}, \frac{2}{3} - \frac{1}{3}i\sqrt{5}\right)$, $C = \begin{bmatrix} \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{\sqrt{5}+1}{\sqrt{10+2\sqrt{5}}} \\ \frac{\sqrt{5}+1}{\sqrt{10+2\sqrt{5}}} & \frac{-2}{\sqrt{10+2\sqrt{5}}} \end{bmatrix}$

g. $D = Diag(3+2i, 3-2i)$, $C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

h. $D = Diag(35i, -30i)$, $C = \begin{bmatrix} \frac{3i}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3i}{\sqrt{13}} \end{bmatrix}$

i. $D = Diag\left(-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} + \left(-\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)i, \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} + \left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)i\right)$,

$$C = \begin{bmatrix} \frac{-1-\sqrt{3}-i-i\sqrt{3}}{2\sqrt{3+\sqrt{3}}} & \frac{-1+\sqrt{3}-i+i\sqrt{3}}{2\sqrt{3+\sqrt{3}}} \\ \frac{2}{2\sqrt{3+\sqrt{3}}} & \frac{2}{2\sqrt{3+\sqrt{3}}} \end{bmatrix}$$

j. $D = Diag(1,-2,2),\ C = \begin{bmatrix} \frac{-1+i}{\sqrt{3}} & \frac{-i}{2\sqrt{3}} & \frac{-i}{2} \\ 0 & \frac{-3}{2\sqrt{3}} & \frac{1}{2} \\ \frac{1}{\sqrt{3}} & \frac{1-i}{2\sqrt{3}} & \frac{1-i}{2} \end{bmatrix}$

k. $D = Diag\left(i\sqrt{5},-i\sqrt{5},i\right),\ C = \begin{bmatrix} \frac{-1+i\sqrt{5}}{2\sqrt{5-\sqrt{5}}} & \frac{-1-i\sqrt{5}}{2\sqrt{5-\sqrt{5}}} & \frac{1}{2} \\ \frac{-1-i\sqrt{5}+2i}{2\sqrt{5-\sqrt{5}}} & \frac{-1+i\sqrt{5}+2i}{2\sqrt{5-\sqrt{5}}} & \frac{1}{2} \\ \frac{1}{\sqrt{5-\sqrt{5}}} & \frac{1}{\sqrt{5-\sqrt{5}}} & \frac{1+i}{2} \end{bmatrix}$

l. $D = Diag\left(\sqrt{5},-\sqrt{5},-1\right),\ C = \begin{bmatrix} \frac{1-2i-i\sqrt{5}}{2\sqrt{5+\sqrt{5}}} & \frac{1-2i+i\sqrt{5}}{2\sqrt{5+\sqrt{5}}} & \frac{1}{2} \\ \frac{1}{\sqrt{5+\sqrt{5}}} & \frac{1}{\sqrt{5+\sqrt{5}}} & \frac{-1-i}{2} \\ \frac{-1-i\sqrt{5}}{\sqrt{5+\sqrt{5}}} & \frac{-1+i\sqrt{5}}{\sqrt{5+\sqrt{5}}} & -\frac{1}{2} \end{bmatrix}$

m. $D = Diag\left(-\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}+\left(-\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\right)i,\ \frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}+\left(\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\right)i,-i\right),$

$$C = \begin{bmatrix} \frac{1}{\sqrt{3+\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} & 0 \\ \frac{-1-\sqrt{3}-\left(1+\sqrt{3}\right)i}{2\sqrt{3+\sqrt{3}}} & \frac{-1+\sqrt{3}-\left(1-\sqrt{3}\right)i}{2\sqrt{3+\sqrt{3}}} & 0 \\ 0 & & 1 \end{bmatrix}$$

n. $D = Diag(5,5+3i,5-3i),\ C = \begin{bmatrix} \frac{2}{3} & \frac{-4-3i}{3\sqrt{10}} & \frac{-4+3i}{3\sqrt{10}} \\ \frac{1}{3} & \frac{-2+6i}{3\sqrt{10}} & \frac{-2-6i}{3\sqrt{10}} \\ \frac{2}{3} & \frac{5}{3\sqrt{10}} & \frac{5}{3\sqrt{10}} \end{bmatrix}$

o. $D = Diag(1,1+7i,1-7i),\ C = \begin{bmatrix} \frac{-6}{7} & \frac{13}{7\sqrt{26}} & \frac{13}{7\sqrt{26}} \\ \frac{2}{7} & \frac{12+21i}{7\sqrt{26}} & \frac{12-21i}{7\sqrt{26}} \\ \frac{3}{7} & \frac{18-14i}{7\sqrt{26}} & \frac{18+14i}{7\sqrt{26}} \end{bmatrix}$

2. Answers:
   a. σ_1 is a real symmetric matrix, σ_2 is Hermitian and σ_3 is diagonal.

b. For σ_1 : $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$;

For σ_2 : $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (also), $U = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$

8. $a + bi = \frac{4}{5} - \frac{3}{5}i$
9. No solutions.
10. $D = \begin{bmatrix} \cos(\theta) + i\sin(\theta) & 0 \\ 0 & \cos(\theta) - i\sin(\theta) \end{bmatrix}$.

    If $\theta = n\pi$ for an integer n, then $U = I_2$,

    otherwise $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$

10.7 Exercises

If you used technology to find a basis for the eigenspaces, the simultaneously diagonalizing matrix that you find may not be the same as the given answer. However, the diagonal matrices should be the same, up to a permutation of the diagonal entries.

1. Answers:

   a. A has only 1-dimensional eigenspaces.

   $C = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$; $C^{-1} = \begin{bmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ -3 & -4 & 2 \end{bmatrix}$

   $C^{-1}AC = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, and $C^{-1}BC = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   b. B has only 1-dimensional eigenspaces.

   $C = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$; $C^{-1} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

   $C^{-1}AC = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and $C^{-1}BC = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

   c. Both A and B have a 2-dimensional eigenspace. Use the eigenspaces of A to find C.

$$C = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}; \; C^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$H = CG = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix};$$

$$H^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$H^{\top}AH = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \text{ and } H^{\top}BH = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

d. Both *A* and *B* have a 2-dimensional eigenspace. Use the eigenspaces of *A* to find *C*.

$$C = \begin{bmatrix} -4 & 1 & -1 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}; \; C^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 2 \\ -3 & 3 & -2 \end{bmatrix}$$

$$H = CG = \begin{bmatrix} -4 & 1 & -1 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -2 & -3 \\ -2 & -1 & -1 \\ 3 & 1 & 2 \end{bmatrix},$$

$$H^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & -2 \\ -1 & 2 & 0 \end{bmatrix}$$

$$C^{-1}AC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \text{ and } C^{-1}BC = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

e. Both are symmetric, but *B* has only 1-dimensional eigenspaces.

$$C = Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{1}{3\sqrt{2}} \\ 0 & -\frac{1}{3} & \frac{4}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{1}{3\sqrt{2}} \end{bmatrix}; \; C^{-1} = Q^{\top} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$$

$$Q^{\top}AQ = \begin{bmatrix} 18 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & 18 \end{bmatrix}, \text{ and } Q^{\top}BQ = \begin{bmatrix} -9 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 45 \end{bmatrix}$$

f. Both matrices are symmetric, and both have a 2-dimensional eigenspace. Use A, with Gram-Schmidt, to find C.

$$C = Q = \begin{bmatrix} -\frac{1}{5}\sqrt{5} & -\frac{4}{15}\sqrt{5} & \frac{2}{3} \\ \frac{2}{5}\sqrt{5} & -\frac{2}{15}\sqrt{5} & \frac{1}{3} \\ 0 & \frac{1}{3}\sqrt{5} & \frac{2}{3} \end{bmatrix};$$

$$C^{-1} = Q^{\top} = \begin{bmatrix} -\frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} & 0 \\ -\frac{4}{15}\sqrt{5} & -\frac{2}{15}\sqrt{5} & \frac{1}{3}\sqrt{5} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$H = QG = \begin{bmatrix} -\frac{1}{5}\sqrt{5} & -\frac{4}{15}\sqrt{5} & \frac{2}{3} \\ \frac{2}{5}\sqrt{5} & -\frac{2}{15}\sqrt{5} & \frac{1}{3} \\ 0 & \frac{1}{3}\sqrt{5} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -\frac{3}{10}\sqrt{10} & \frac{1}{10}\sqrt{10} & 0 \\ \frac{1}{10}\sqrt{10} & \frac{3}{10}\sqrt{10} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6}\sqrt{2} & -\frac{1}{2}\sqrt{2} & \frac{2}{3} \\ -\frac{2}{3}\sqrt{2} & 0 & \frac{1}{3} \\ \frac{1}{6}\sqrt{2} & \frac{1}{2}\sqrt{2} & \frac{2}{3} \end{bmatrix}, \text{ with } H^{-1} = H^{\top}$$

$$H^{\top}AH = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \text{ and } H^{\top}BH = \begin{bmatrix} -27 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 45 \end{bmatrix}$$

g. Both are symmetric, and both have a 2-dimensional eigenspace. Use the eigenspaces of A along with Gram-Schmidt to find C.

$$C = Q = \begin{bmatrix} \frac{1}{6}\sqrt{6} & -\frac{2}{5}\sqrt{5} & -\frac{1}{30}\sqrt{30} \\ \frac{1}{3}\sqrt{6} & \frac{1}{5}\sqrt{5} & -\frac{1}{15}\sqrt{30} \\ \frac{1}{6}\sqrt{6} & 0 & \frac{1}{6}\sqrt{30} \end{bmatrix};$$

$$C^{-1} = Q^{\top} = \begin{bmatrix} \frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ -\frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ -\frac{1}{30}\sqrt{30} & -\frac{1}{15}\sqrt{30} & \frac{1}{6}\sqrt{30} \end{bmatrix}$$

$$H = CG = \begin{bmatrix} \frac{1}{6}\sqrt{6} & -\frac{2}{5}\sqrt{5} & -\frac{1}{30}\sqrt{30} \\ \frac{1}{3}\sqrt{6} & \frac{1}{5}\sqrt{5} & -\frac{1}{15}\sqrt{30} \\ \frac{1}{6}\sqrt{6} & 0 & \frac{1}{6}\sqrt{30} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5}\sqrt{10} & -\frac{1}{5}\sqrt{15} \\ 0 & \frac{1}{5}\sqrt{15} & \frac{1}{5}\sqrt{10} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6}\sqrt{6} & -\frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & 0 & -\frac{1}{3}\sqrt{3} \\ \frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2} & \frac{1}{3}\sqrt{3} \end{bmatrix}, \text{ with } H^{-1} = H^{\top}$$

$$H^{\top}AH = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \text{ and } H^{\top}BH = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

h. A has only 1-dimensional eigenspaces. Neither is symmetric.

$$C = \begin{bmatrix} -1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ -4 & -1 & 0 & -2 \\ 1 & 1 & 1 & 2 \end{bmatrix}; \ C^{-1} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 4 & 2 & -3 & -8 \\ 3 & 0 & -2 & -5 \\ -4 & -1 & 3 & 8 \end{bmatrix}$$

$$C^{-1}AC = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \text{ and } C^{-1}BC = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

i. Both matrices are symmetric, and both have two 2-dimensonal eigenspaces. Use the eigenspaces of A, along with Gram-Schmidt.

$$Q = \begin{bmatrix} \frac{2}{129}\sqrt{129} & \frac{8}{1419}\sqrt{2838} & -\frac{7}{69}\sqrt{69} & -\frac{8}{759}\sqrt{1518} \\ \frac{2}{129}\sqrt{129} & -\frac{9}{946}\sqrt{2838} & -\frac{4}{69}\sqrt{69} & \frac{9}{506}\sqrt{1518} \\ \frac{11}{129}\sqrt{129} & \frac{1}{1419}\sqrt{2838} & \frac{2}{69}\sqrt{69} & -\frac{1}{759}\sqrt{1518} \\ 0 & \frac{1}{66}\sqrt{2838} & 0 & \frac{1}{66}\sqrt{1518} \end{bmatrix}$$

$$G = \begin{bmatrix} \frac{2}{129}\sqrt{129} & \frac{8}{1419}\sqrt{2838} & -\frac{7}{69}\sqrt{69} & -\frac{8}{759}\sqrt{1518} \\ \frac{2}{129}\sqrt{129} & -\frac{9}{946}\sqrt{2838} & -\frac{4}{69}\sqrt{69} & \frac{9}{506}\sqrt{1518} \\ \frac{11}{129}\sqrt{129} & \frac{1}{1419}\sqrt{2838} & \frac{2}{69}\sqrt{69} & -\frac{1}{759}\sqrt{1518} \\ 0 & \frac{1}{66}\sqrt{2838} & 0 & \frac{1}{66}\sqrt{1518} \end{bmatrix}$$

$$H = QG = \begin{bmatrix} \frac{2}{33}\sqrt{33} & 0 & \frac{1}{11}\sqrt{66} & -\frac{1}{3}\sqrt{3} \\ -\frac{2}{33}\sqrt{33} & -\frac{1}{6}\sqrt{6} & \frac{5}{66}\sqrt{66} & \frac{1}{3}\sqrt{3} \\ \frac{1}{11}\sqrt{33} & -\frac{1}{3}\sqrt{6} & -\frac{1}{33}\sqrt{66} & 0 \\ \frac{4}{33}\sqrt{33} & \frac{1}{6}\sqrt{6} & \frac{1}{66}\sqrt{66} & \frac{1}{3}\sqrt{3} \end{bmatrix}$$

$$H^{-1}AH = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 66 & 0 \\ 0 & 0 & 0 & 66 \end{bmatrix}, \text{ and } H^{-1}BH = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & 24 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 24 \end{bmatrix}$$

j. Both are not symmetric, and both have a 2-dimensional eigenspace. Use the eigenspaces of A along with Gram-Schmidt to find C.

$$H = CG = \begin{bmatrix} -2 & -1 & 0 & -2 \\ -2 & -2 & -1 & -3 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & 0 & -2 \\ -2 & -2 & -1 & -2 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \text{ and } H^{-1} = \begin{bmatrix} -2 & 1 & 1 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -2 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

$$H^{-1}AH = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \text{ and } H^{-1}BH = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

k. Both are symmetric, with A having a 3-dimensional eigenspace and B having two 2-dimensional eigenspaces. Use the eigenspaces of A along with Gram-Schmidt.

$$H = QG = \begin{bmatrix} \frac{1}{15}\sqrt{15} & \frac{1}{2}\sqrt{2} & -\frac{3}{22}\sqrt{22} & -\frac{2}{165}\sqrt{165} \\ -\frac{1}{15}\sqrt{15} & \frac{1}{2}\sqrt{2} & \frac{3}{22}\sqrt{22} & \frac{2}{165}\sqrt{165} \\ \frac{1}{5}\sqrt{15} & 0 & \frac{1}{11}\sqrt{22} & -\frac{2}{55}\sqrt{165} \\ \frac{2}{15}\sqrt{15} & 0 & 0 & \frac{1}{15}\sqrt{165} \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{6}\sqrt{30} & \frac{1}{6}\sqrt{3} & -\frac{1}{6}\sqrt{3} \\ 0 & -\frac{1}{66}\sqrt{330} & \frac{1}{6}\sqrt{33} & \frac{1}{66}\sqrt{33} \\ 0 & \frac{1}{11}\sqrt{11} & 0 & \frac{1}{11}\sqrt{110} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{15}\sqrt{15} & \frac{1}{5}\sqrt{15} & -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} \\ -\frac{1}{15}\sqrt{15} & \frac{2}{15}\sqrt{15} & \frac{1}{3}\sqrt{6} & 0 \\ \frac{1}{5}\sqrt{15} & -\frac{1}{15}\sqrt{15} & \frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} \\ \frac{2}{15}\sqrt{15} & \frac{1}{15}\sqrt{15} & 0 & \frac{1}{3}\sqrt{6} \end{bmatrix}, \text{ with } H^{-1} = H^{T};$$

$$H^{\top}AH = \begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \text{ and } H^{\top}BH = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

l. Both are symmetric, each with a 2-dimensional and a 3-dimensional eigenspace. Use the eigenspaces of A along with Gram-Schmidt to find Q.

$$H = QG = \begin{bmatrix} -\frac{1}{9}\sqrt{3} & -\frac{2}{45}\sqrt{3}\sqrt{5} & -\frac{1}{6}\sqrt{3} & -\frac{1}{5}\sqrt{15} & \frac{1}{2} \\ -\frac{1}{9}\sqrt{3} & \frac{7}{45}\sqrt{3}\sqrt{5} & \frac{1}{6}\sqrt{3} & -\frac{2}{15}\sqrt{15} & -\frac{1}{2} \\ \frac{5}{9}\sqrt{3} & \frac{1}{45}\sqrt{3}\sqrt{5} & 0 & -\frac{1}{15}\sqrt{15} & 0 \\ 0 & \frac{1}{5}\sqrt{3}\sqrt{5} & -\frac{1}{6}\sqrt{3} & \frac{1}{15}\sqrt{15} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2}\sqrt{3} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\cdot \begin{bmatrix} -\frac{2}{3} & \frac{2}{9}\sqrt{5} & -\frac{5}{9} & 0 & 0 \\ \frac{1}{3}\sqrt{5} & \frac{4}{9} & -\frac{2}{9}\sqrt{5} & 0 & 0 \\ 0 & \frac{1}{3}\sqrt{5} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{10}\sqrt{15} & 0 & -\frac{1}{5}\sqrt{15} & \frac{1}{2} \\ \frac{1}{3}\sqrt{3} & \frac{1}{10}\sqrt{15} & 0 & -\frac{2}{15}\sqrt{15} & -\frac{1}{2} \\ -\frac{1}{3}\sqrt{3} & \frac{2}{15}\sqrt{15} & -\frac{1}{3}\sqrt{3} & -\frac{1}{15}\sqrt{15} & 0 \\ \frac{1}{3}\sqrt{3} & \frac{1}{30}\sqrt{15} & -\frac{1}{3}\sqrt{3} & \frac{1}{15}\sqrt{15} & \frac{1}{2} \\ 0 & \frac{1}{6}\sqrt{15} & \frac{1}{3}\sqrt{3} & 0 & \frac{1}{2} \end{bmatrix}, \text{ with } H^{-1} = H^{\top};$$

$$H^{\top}AH = \begin{bmatrix} -60 & 0 & 0 & 0 & 0 \\ 0 & -60 & 0 & 0 & 0 \\ 0 & 0 & -60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and}$$

$$H^{\top}BH = \begin{bmatrix} -40 & 0 & 0 & 0 & 0 \\ 0 & -40 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 20 \end{bmatrix}$$

m. Both are symmetric; A has a 4-dimensional eigenspace, and B has a 3-dimensional and a 2-dimensional eigenspace.

$H = QG =$

$$\begin{bmatrix} -\frac{2}{29}\sqrt{29} & -\frac{1}{174}\sqrt{29}\sqrt{30} & -\frac{1}{186}\sqrt{30}\sqrt{31} & -\frac{2}{217}\sqrt{31}\sqrt{35} & \frac{1}{7}\sqrt{5}\sqrt{7} \\ \frac{5}{29}\sqrt{29} & -\frac{1}{435}\sqrt{29}\sqrt{30} & -\frac{1}{465}\sqrt{30}\sqrt{31} & -\frac{4}{1085}\sqrt{31}\sqrt{35} & \frac{2}{35}\sqrt{5}\sqrt{7} \\ 0 & \frac{1}{30}\sqrt{29}\sqrt{30} & -\frac{1}{930}\sqrt{30}\sqrt{31} & -\frac{2}{1085}\sqrt{31}\sqrt{35} & \frac{1}{35}\sqrt{5}\sqrt{7} \\ 0 & 0 & \frac{1}{31}\sqrt{30}\sqrt{31} & -\frac{2}{1085}\sqrt{31}\sqrt{35} & \frac{1}{35}\sqrt{5}\sqrt{7} \\ 0 & 0 & 0 & \frac{1}{35}\sqrt{31}\sqrt{35} & \frac{2}{35}\sqrt{5}\sqrt{7} \end{bmatrix}$$

$$\cdot \begin{bmatrix} -\frac{1}{362}\sqrt{52\,490} & -\frac{35}{10\,498}\sqrt{10\,498} & 0 & \frac{1}{29}\sqrt{406} & 0 \\ 0 & \frac{1}{435}\sqrt{78\,735} & -\frac{1}{30}\sqrt{435} & \frac{1}{174}\sqrt{3045} & 0 \\ 0 & \frac{1}{465}\sqrt{84\,165} & \frac{1}{30}\sqrt{465} & \frac{1}{186}\sqrt{3255} & 0 \\ \frac{1}{362}\sqrt{78\,554} & -\frac{5}{11\,222}\sqrt{392\,770} & 0 & \frac{1}{31}\sqrt{310} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{7}\sqrt{14} & \frac{1}{7}\sqrt{35} \\ -\frac{29}{1810}\sqrt{1810} & -\frac{7}{362}\sqrt{362} & 0 & \frac{1}{7}\sqrt{14} & \frac{2}{35}\sqrt{35} \\ -\frac{1}{905}\sqrt{1810} & \frac{6}{181}\sqrt{362} & -\frac{1}{2}\sqrt{2} & \frac{1}{14}\sqrt{14} & \frac{1}{35}\sqrt{35} \\ -\frac{1}{905}\sqrt{1810} & \frac{6}{181}\sqrt{362} & \frac{1}{2}\sqrt{2} & \frac{1}{14}\sqrt{14} & \frac{1}{35}\sqrt{35} \\ \frac{31}{1810}\sqrt{1810} & -\frac{5}{362}\sqrt{362} & 0 & \frac{1}{7}\sqrt{14} & \frac{2}{35}\sqrt{35} \end{bmatrix},$$

with $H^{-1} = H^{\mathsf{T}}$;

$$H^{\mathsf{T}}AH = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 35 \end{bmatrix}, \text{ and } H^{\mathsf{T}}BH = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

n. Neither matrix is symmetric. A has two 3-dimensional eigenspaces, while B has a 4-dimensional and a 2-dimensional eigenspace.

$$H = CG = \begin{bmatrix} -54 & 6 & 4 & 26 & -46 & -25 \\ 17 & -5 & 2 & 4 & -2 & 19 \\ 26 & -2 & -4 & -100 & 50 & 20 \\ 8 & 0 & 0 & 33 & 0 & 0 \\ 0 & 8 & 0 & 0 & 33 & 0 \\ 0 & 0 & 8 & 0 & 0 & 33 \end{bmatrix} \begin{bmatrix} \frac{3}{11} & \frac{2}{11} & \frac{1}{15} & 0 & 0 & 0 \\ 1 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{96}{11} & -\frac{64}{11} & -\frac{8}{5} & 26 & -46 & -25 \\ -\frac{4}{11} & \frac{56}{11} & \frac{24}{5} & 4 & -2 & 19 \\ \frac{56}{11} & \frac{8}{11} & -\frac{8}{5} & -100 & 50 & 20 \\ \frac{24}{11} & \frac{16}{11} & \frac{8}{15} & 33 & 0 & 0 \\ 8 & 0 & -\frac{8}{3} & 0 & 33 & 0 \\ 0 & 8 & 8 & 0 & 0 & 33 \end{bmatrix};$$

$$H^{-1} = \begin{bmatrix} 5 & \frac{53}{4} & \frac{3}{2} & -1 & \frac{11}{2} & -\frac{19}{4} \\ \frac{17}{8} & 9 & \frac{1}{2} & -\frac{5}{4} & \frac{11}{4} & -\frac{31}{8} \\ -\frac{15}{8} & -\frac{15}{4} & -\frac{15}{8} & -\frac{15}{4} & 0 & \frac{15}{8} \\ -\frac{13}{33} & -\frac{40}{33} & -\frac{1}{11} & \frac{7}{33} & -\frac{16}{33} & \frac{5}{11} \\ -\frac{15}{11} & -\frac{116}{33} & -\frac{17}{33} & -\frac{2}{33} & -\frac{43}{33} & \frac{43}{33} \\ -\frac{2}{33} & -\frac{14}{11} & \frac{1}{3} & \frac{40}{33} & -\frac{2}{3} & \frac{17}{33} \end{bmatrix}$$

$$H^{-1}AH = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } H^{-1}BH = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$