In this chapter we will be working with inverse functions. Dictionaries define inverse as “something that is opposite or reversed in order.” Similarly, the inverse of a function is simply the set of ordered pairs obtained when we swap the input and output values in the original function. Now, how does this apply to the real world?

We are all familiar with the traditional phone books. Yes... those heavy phone books that probably end up in the recycling bin. In today’s technologically enhanced world, to find a telephone number we simply do an online search, or use our cell phones. Moreover, we can now conduct an online reverse phone search. That is, we can type a telephone number and find the person or address associated with it. Such an inverse finding was not possible with the traditional phone book. With the invention of the smartphone, doing a reverse search is even possible without being at a computer. This reverse phone search is an example of an inverse function.

Most everyone has heard of the company Motorola, but how many of us know of the history of the cell phone and the person who invented it? In addition to the many inverse function applications that we will solve in this chapter, we will also learn about the relationship between Motorola and the inventor of the cell phone, and how the company chose their name. We will also discover some fascinating facts about many other interesting inverse functions associated with other common real-life scenarios.

Section 5.0

Getting Started

This section is intended as a review of concepts and techniques that will facilitate your work with the content to be addressed in Chapter 5. Here we will help you review solving literal equations and evaluating functions.

Literal Equations

An equation that expresses the relationship between two or more letters representing variables or constants is called a literal equation. A literal equation differs from other equations in that we are not solving for a numeric value for a specific variable, but solving for one letter in particular.
A formula is an example of a literal equation. For example, we know that we can calculate the distance we travel from one location to another with the formula $d = rt$, where $d$ represents distance, $r$ is the rate or constant speed, and $t$ equals time. So, the distance traveled moving at a constant speed is the product of the rate and the time traveled. But, what if we already know the distance traveled and the time but we wish to determine the speed? In this case we would be solving the equation for $r$.

### Solving a Literal Equation

We will solve a literal equation for one of the variables. Apply the same rules and steps used for solving other equations:

- Isolate the desired variable (leave it by itself on one side of the equation) by applying inverse operations.
- Whatever operations you perform on one side of the equation, you must execute on the other side.
- If the equation contains fractions, first clear the fractions by multiplying both sides of the equation by the least common denominator (LCD).

**Little Facts:** Why are they called literal equations? The word literal comes from the Latin “littera” which means letter. So, a literal equation is just an equation with several different letters. Latin is an ancient language that it is officially adopted only in the Vatican State in Italy; it has no native speakers. Source: www.latinlanguage.org

**EXAMPLE 1**

Solve each equation for the specified variable.

a. $x = y^3 + 8$, for $y$

b. $s = \frac{2}{t + 8}$, for $t$

c. $3x = 5y + 9$, for $y$

d. $3n = \frac{mr}{s^2}$, for $s$

**Solution**

a. To solve for $y$, we begin by subtracting 8 from both sides.

$$x = y^3 + 8 \rightarrow x - 8 = y^3$$

To eliminate the cube, we will now take the cube root on both sides of the equation.

$$\sqrt[3]{y^3} = \sqrt[3]{x - 8} \rightarrow y = \sqrt[3]{x - 8}$$

b. To solve the equation for $t$, the first step is to multiply both sides by the LCD.

$$(t + 8)s = \frac{2}{t + 8} (t + 8)$$

$$(t + 8)s = 2$$

Now we solve the resulting equation for $t$. We can do this by following one of two approaches.

$$(t + 8)s = 2 \rightarrow t + 8 = \frac{2}{s} \rightarrow t = \frac{2}{s} - 8$$

or $$(t + 8)s = 2 \rightarrow ts + 8s = 2 \rightarrow t = \frac{2 - 8s}{s} = \frac{2}{s} - 8$$

c. To isolate $y$, we will subtract 9 from both sides, and then divide both sides by 5.

$$3x = 5y + 9 \rightarrow 5y = 3x - 9 \rightarrow y = \frac{3x - 9}{5}$$

We can also write the answer as $y = \frac{3}{5}x - \frac{9}{5}$. 
d. The first step is to multiply both sides by $s^2$.

$$3n = \frac{mr}{s^2} \implies 3ns^2 = mr$$

Then we divide both sides by $3n$ to isolate $s^2$, and finally take the square root on both sides to solve for $s$.

$$s^2 = \frac{mr}{3n} \implies s = \sqrt{\frac{mr}{3n}}$$

### Evaluating Functions

Recall that $y = f(x)$ states that $y$ is a function of $x$, and $f(x)$ means the value of $f$ for specific values of $x$. In this chapter, you will be asked to evaluate many functions. It is important to remember that evaluating a function implies finding an output value for a specific value of the input. Let us review this competency.

#### EXAMPLE 2

Evaluate the following functions at the given values of the input. Express your answer in function notation and in ordered pair format.

a. $f(x) = 2x^4 - 9x^3 + 1$, find $f(-1)$

b. $h(x) = -3\sqrt{x + 12} + 5x$, find $h(0)$

c. $g(x) = -|x - 6|$, find $g(-3)$

**Solution**

a. $f(-1)$ means that we will replace every $x$ we see in the function with $-1$.

$$f(-1) = 2(-1)^4 - 9(-1)^3 + 1 = 2(1) - 9(-1) + 1 = 12$$

Therefore, $f(-1) = 12$, which represents the ordered pair $(-1, 12)$.

b. $h(0) = -3\sqrt{0 + 12} + 5(0) = -3\sqrt{12} + 0 = -3\sqrt{4 \cdot 3} = -3(2)\sqrt{3} = -6\sqrt{3}$.

So, $h(0) = -6\sqrt{3}$, or equivalently, $(0, -6\sqrt{3})$.

c. $g(-3) = -|-3 - 6| = -9 = -9$.

Thus, $g(-3) = -9$; in ordered pair format we have $(-3, -9)$.

#### EXAMPLE 3

Evaluate $f(x) = x^2 - 3x + 1$ for the given inputs.

a. $f\left(\frac{1}{4}\right)$

b. $f(a)$

c. $f(x + h)$

d. $f(x^2 - 3x + 1)$

**Solution**

a. $f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 - 3\left(\frac{1}{4}\right) + 1 = \frac{1}{16} - \frac{3}{4} + 1 = \frac{1}{16} - \frac{12}{16} + \frac{16}{16} = \frac{5}{16}$

b. $f(a) = a^2 - 3a + 1$

c. $f(x + h) = (x + h)^2 - 3(x + h) + 1 = x^2 + 2hx + h^2 - 3x - 3h + 1$

d. Observe that to find $f(x^2 - 3x + 1)$ we will replace each $x$ in the given function with the expression $x^2 - 3x + 1$. This is like evaluating the function when the input is itself.

$$f(x^2 - 3x + 1) = (x^2 - 3x + 1)^2 - 3(x^2 - 3x + 1) + 1$$

$$= (x^2 - 3x + 1)(x^2 - 3x + 1) - 3(x^2 - 3x + 1) + 1$$

$$= x^4 - 6x^3 + 11x^2 - 6x + 1 - 3x^2 + 9x - 3 + 1$$

$$= x^4 - 6x^3 + 8x^2 + 3x - 1$$
EXAMPLE 4

Given \( f(a) = \frac{4}{a - 2} \), find \( f\left(\frac{4}{a} + 2\right) \).

Solution

We will replace \( a \) in the given function with the expression \( \left(\frac{4}{a} + 2\right) \).

\[
\begin{align*}
f\left(\frac{4}{a} + 2\right) &= \frac{4}{\left(\frac{4}{a} + 2\right) - 2} \\
&= \frac{4}{\frac{4}{a} + 0} \\
&= \frac{4}{\frac{4}{a}} \\
&= 4 \cdot \frac{a}{4} = a.
\end{align*}
\]

The “\( a \)” in the original function has been replaced with this expression.

Reviewing how to evaluate functions at different outputs will facilitate your work with the material you will learn in this chapter.

5.0 Exercises

In exercises 1–12, solve each equation for the specified variable.

1. \( A = \frac{1}{2}bh \), for \( h \)
2. \( y = -5x + 10 \), for \( x \)
3. \( -3x = -14 - 6y \), for \( y \)
4. \( 4x^2 + 1 = 3y \), for \( x \)
5. \( 4x = 9 - 16y^2 \), for \( y \)
6. \( 4R + 9p^2 = 4 \), for \( p \)
7. \( 8c^3 - 4x = -1 \), for \( c \)
8. \( 7 = y + 27r^2 \), for \( r \)
9. \( h = \frac{7}{t - 3} \), for \( t \)
10. \( \frac{12}{2y - 4} = x \), for \( y \)
11. \( \frac{-3k - 1}{15} = b \), for \( k \)
12. \( y = 4 - 9x \), for \( x \)

In exercises 13–24, evaluate the following functions at the given values of the input. Express your answer in function notation and in ordered pair format.

13. \( f(x) = 2x - 5 \), find \( f\left(\frac{2}{3}\right) \)
14. \( g(x) = \frac{1}{2}x^3 - 4x - 1 \), find \( g(-2) \)
15. \( f(x) = 2x^2 - 3x + 4 \), find \( f(-4) \)
16. \( R(x) = 3x^2 - 5 + 2x \), find \( R(-3) \)
17. \( f(x) = \sqrt{2x - 8} \), find \( f(6) \)
18. \( f(x) = \sqrt{-8 + 8x} \), find \( f(3) \)
19. \( f(x) = -\sqrt{2x^2 - 41 + x} \), find \( f(-5) \)
20. \( f(x) = \sqrt{-3x^3 + 28 + 2x} \), find \( f(2) \)
21. \( f(x) = -\sqrt{7x - 3} \), find \( f(-13) \)
22. \( n(x) = 1 - 2|21 - 5x| \), find \( n(10) \)
23. \( q(x) = \frac{3x}{4x^2 - 9} \), find \( q(7) \)
24. \( h(x) = \frac{6 - x}{2x + 1} \), find \( h(-1) \)

In exercises 25–27, evaluate the following functions at the given inputs.

25. \( f(x) = 2x^2 - 5x \)
   a. \( f\left(\frac{3}{2}\right) \)
   b. \( f\left(-\frac{2}{3}\right) \)
   c. \( f(a + 3) \)
   d. \( f(x + h) \)
   e. \( f(x^2 + 4x) \)

26. \( f(x) = -3x^2 + 2x \)
   a. \( f\left(-\frac{1}{4}\right) \)
   b. \( f\left(\frac{2}{5}\right) \)
   c. \( f(a - 2) \)
   d. \( f(x + h) \)
   e. \( f(2x^2 - x) \)
27. \( f(x) = x^2 - x - 5 \)
   a. \( f\left(-\frac{4}{3}\right) \)
   b. \( f(a) \)
   c. \( f(a + 7) \)
   d. \( f(x + h) \)
   e. \( f(3x^2 - 2) \)

---

**ANSWER KEY 5.0 Exercises**

1. \( h = \frac{2A}{b} \)
2. \( x = \frac{10 - y}{5} \) or \( x = 2 - \frac{y}{5} \)
3. \( y = \frac{3x - 14}{6} \) or \( y = \frac{1}{2}x - \frac{7}{3} \)
4. \( x = \pm \frac{\sqrt{3y - 1}}{2} \)
5. \( y = \pm \frac{\sqrt{9 - 4x}}{4} \)
6. \( p = \pm \frac{2}{3} \sqrt{1 - R} \)
7. \( c = \frac{\sqrt{4x - 1}}{2} \)
8. \( r = \frac{\sqrt{7 - y}}{3} \)
9. \( t = 3 + \frac{7}{h} \)
10. \( y = 2 + \frac{6}{x} \)
11. \( k = \frac{-1 - 15b}{3} \) or \( k = -\frac{1}{3} - 5b \)
12. \( x = \frac{4 - 5y}{9} \) or \( x = \frac{4}{9} - \frac{5}{9}y \)
13. \( f\left(\frac{2}{3}\right) = -\frac{11}{3} \left( \frac{2}{3} - \frac{11}{3} \right) \)
14. \( g(-2) = 3, (-2, 3) \)
15. \( f(-4) = 48, (-4, 48) \)
16. \( R(-3) = 16, (-3, 16) \)
17. \( f(6) = 2, (6, 2) \)
18. \( f(3) = 4, (3, 4) \)
19. \( f(-5) = -8, (-5, -8) \)
20. \( f(2) = 6, (2, 6) \)
21. \( f(-13) = -94, (-13, -94) \)
22. \( n(10) = -57, (10, -57) \)
23. \( q(7) = \left( \frac{21}{19}, \frac{21}{19} \right) \)
24. \( h(1) = 7, (-1, 7) \)
25. a. 3
   b. \( \frac{38}{9} \)
   c. \( 2a^2 + 7a + 3 \)
   d. \( 2x^2 + 4xh + 2h^2 - 5x - 5h \)
   e. \( 2x^4 + 16x^3 + 27x^2 - 20x \)
26. a. \( -\frac{11}{16} \)
   b. \( \frac{8}{25} \)
   c. \( -3a^2 + 16a - 16 \)
   d. \( -3x^2 - 6xh + 3h^2 + 2x + 2h \)
   e. \( -12x^4 + 12x^3 + x^2 - 2x \)
27. a. \( -\frac{17}{9} \)
   b. \( a^2 - a - 5 \)
   c. \( a^2 + 13a + 37 \)
   d. \( x^2 + 2xh + h^2 - x - h - 5 \)
   e. \( 9x^3 - 15x^2 + 1 \)

---

**Section 5.1 Finding the Inverse: Numerical, Graphical, and Symbolic Approaches**

Consider the data from the following two tables. Does each table represent a function?

<table>
<thead>
<tr>
<th>a. Input</th>
<th>-1</th>
<th>-2</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>2</td>
<td>0</td>
<td>14</td>
<td>18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>b. Input</th>
<th>-3</th>
<th>-4</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>9</td>
<td>16</td>
<td>9</td>
<td>36</td>
</tr>
</tbody>
</table>

Yes. In each table, we can see that for each input there is only one output. How do tables (c) and (d) that follow compare to tables (a) and (b), respectively? What has happened?
If you noticed that we have switched the inputs and the outputs, you are correct! Thus, we have reversed the roles of the independent and dependent variables. A more important question arises: Does the second set of tables represent functions? Table (c) is a function, but observe that table (d) is not, because the input 9 has two different outcomes.

When we reverse the action of a function, as we have done with our table examples, we are looking for the inverse of the function. But, what is the meaning of inverse? In everyday life, it means something that is opposite or reversed in order. Think of the action of tying your shoelaces; the inverse process would be untying the shoelaces. Or, you may use your computer to find someone’s telephone number. You input the name, and the computer provides you with the corresponding number. The inverse of this action would be to input a telephone number and have the computer search for its owner.

Similarly, the inverse of a function is simply the set of ordered pairs obtained when we swap the input, \(x\), and the output, \(y\), coordinates in the original function. Just a caveat here, when we interchange the independent and dependent variables of a function the newly formed relation may not turn out to be a function, just as we observed in our example!

So, how can we guarantee that the inverse of the original function will also be a function? The original function must be a one-to-one function. Since we are interested in working with inverses that are functions, let us first understand the definition of a one-to-one function.

**One-to-One Function**

A function \(f\) is one-to-one if for any elements \(x_1\) and \(x_2\) in its domain, when \(x_1 \neq x_2\) then \(f(x_1) \neq f(x_2)\).

That is, any two different inputs will always produce two different outputs.

Notice that we use the terminology one-to-one because, just as in a function for every \(x\) there is exactly one \(y\), now for every \(y\) there is exactly one \(x\).

**EXAMPLE 1**

Determine whether each function is one-to-one.

a. \(f(x) = x^3 + 2\)  
b. \(f(x) = x^2 - 8\)

**Solution**

a. The function will be one-to-one if no two distinct \(x\)-values produce the same \(y\)-value. Let us test several different inputs and check their outputs.

<table>
<thead>
<tr>
<th>Input</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>-6</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

Observe that different \(x\)-values produced different \(y\)-values. For this function, in general, if \(x_1 \neq x_2\), then \((x_1)^3 \neq (x_2)^3\) and we can say that \((x_1)^3 + 2 \neq (x_2)^3 + 2\). Thus, this function is one-to-one.

b. Let \(x_1 = -3\) and \(x_2 = 3\). Then \(f(x) = x^2 - 8\) will produce the same output for these two distinct inputs. Example:

\[f(-3) = (-3)^2 - 8 = 9 - 8 = 1\]  
\[f(3) = (3)^2 - 8 = 9 - 8 = 1\]

The given function is not one-to-one.

Showing that a function is one-to-one is often simpler if we use a graphical method. Just as we have used the vertical line test to determine whether a given graph represents a function, we can now use a **horizontal line test** to determine from its graph whether a function is one-to-one.
5.1 Finding the Inverse: Numerical, Graphical, and Symbolic Approaches

**Horizontal Line Test**

A function is one-to-one if no horizontal line intersects its graph more than once.

Let us apply this visual method to support the answers to the previous exercises.

**EXAMPLE 2**

Use the horizontal line test to determine whether each function is one-to-one.

a. $f(x) = x^3 + 2$  
   b. $f(x) = x^2 - 8$

**Solution**

a. The graph of $f(x) = x^3 + 2$ is shown, along with a horizontal line.

Observe that the horizontal line intersects the graph of $f(x) = x^3 + 2$ only once. In fact, no horizontal line would intersect the graph of this function more than once. $f(x) = x^3 + 2$ is a one-to-one function.

b. Notice how the graph of $f(x) = x^2 - 8$ does not pass the horizontal line test.

For example, examine the $y$-coordinate for $x = -3$ and $x = 3$. When $x = -3$, $y = 1$, and for $x = 3$, $y = 1$. Since we have two distinct $x$-values producing the same $y$-value, the given function is not one-to-one.
EXAMPLE 3
Which of the following represent one-to-one functions?

a. \( y = \sqrt{3x + 6} \)  
   b. \( f = \{(-1, 6), (0, -5), (8, 1), (4, -5), (3, 11)\} \)  
   c. \( y = -2x^2 \)  
   d. \( y = 3x + 6 \)

**Solution**

a. The graph of \( y = \sqrt{3x + 6} \) is shown.

![Graph of \( y = \sqrt{3x + 6} \)](image)

The horizontal line demonstrates that the function is one-to-one.

b. This function is not one-to-one because we have two different inputs producing the same output. The \( y \)-coordinate is the same for the ordered pairs \((0, -5)\) and \((4, -5)\).

c. The graph of \( y = -2x^2 \) is a concave down parabola, thus a horizontal line would intersect the graph of the function more than once. The function is not one-to-one.

d. The graph of \( y = 3x + 6 \) is an increasing line and no horizontal line would intersect the graph of this function more than once. This function is one-to-one.

Note: Increasing or decreasing lines will pass the horizontal line test, and are one-to-one functions. A horizontal line will clearly fail the horizontal line test, and recall that a vertical line is not a function.

**Inverse Functions**

We are now ready to expand the concept of inverse functions. As we pointed out previously, the inverse function is what we call an “undoing” function. Rita, a bank teller in Montgomery, AL, has a function to help her calculate her salary for the amount of hours of work in a week. If she earns \$10.26\) per hour, she only needs to multiply the hourly pay times the number of hours worked. Thus, if Rita works 20 hours, her pay will be \$10.26(20) = \$205.20. So, Rita’s function for her weekly pay, \( p \), is \( p(h) = 10.26h \) for \( h \) hours worked.

A sample table of values follows for different hours of work:

<table>
<thead>
<tr>
<th>Hours</th>
<th>Pay in Dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>102.60</td>
</tr>
<tr>
<td>20</td>
<td>205.20</td>
</tr>
<tr>
<td>22</td>
<td>225.72</td>
</tr>
<tr>
<td>35</td>
<td>359.10</td>
</tr>
</tbody>
</table>

Now, suppose we know that Rita earned \$225.72. How do we calculate how many hours she worked that week? The “undoing” or inverse function to answer this question would be to divide the pay by the hourly rate: \( 225.72/10.26 = 22 \). So, our function for the weekly hours of work, \( h \), is \( h(p) = p/10.26 \) for \( p \) dollars earned.

<table>
<thead>
<tr>
<th>Pay in Dollars</th>
<th>102.60</th>
<th>205.20</th>
<th>225.72</th>
<th>359.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hours</td>
<td>10</td>
<td>20</td>
<td>22</td>
<td>35</td>
</tr>
</tbody>
</table>
Notice that the pay function, \( p(h) \), is one-to-one and the inverse function, \( h(p) \), also one-to-one, can be obtained by simply swapping the inputs and the outputs from the table of values in the original function.

### Inverse Function

If \( f \) is a one-to-one function with ordered pairs \((x, y)\), the **inverse of \( f \)**, denoted \( f^{-1} \), is also a one-to-one function with ordered pairs \((y, x)\).

That is, the inverse of a function is the set of ordered pairs obtained when we swap the inputs and the outputs in the original function.

The domain of \( f^{-1} \) is the same as the range of \( f \), and the range of \( f^{-1} \) is the same as the domain of \( f \).

\( f^{-1} \) is read as “\( f \) inverse.”

### CAUTION

Do not treat \( f^{-1}(x) \) as an expression having the exponent negative one. \( f^{-1}(x) \) is the notation for the inverse function and it does not mean the reciprocal of \( f(x) \). That is, \( f^{-1}(x) \neq \frac{1}{f(x)} \).

As we have learned, if a function is not one-to-one, it will not have an inverse function. For the most part, we are interested in working with functions whose inverses are also functions.

### EXAMPLE 4

Determine if each function is one-to-one. If so, find its inverse function.

**a.** \( f = \{(5, -3), (-2, 4), (1, 0), (-1, -7)\} \)

**b.** \( f = \{(0, 6), (-2, -3), (1, -5), (4, 6)\} \)

**c.** \( y = -2x + 8 \)

**Solution**

**a.** Since different \( x \)-values produce different \( y \)-values, we only need to interchange the \( x \)- and \( y \)-coordinates:

\[ f^{-1} = \{(-3, 5), (4, -2), (0, 1), (-7, -1)\} \]

**b.** This function is not one-to-one; two ordered pairs share the same \( y \)-coordinate 6. Thus, the given function has no inverse function.

**c.** \( y = -2x + 8 \) is a decreasing line and passes the horizontal line test. To find the inverse we only need to interchange the \( x \) and \( y \) variables, and solve for “the new” \( y \).

\[
\begin{align*}
  y &= -2x + 8 \\
  x &= -2y + 8 \quad \text{Interchange the } x \text{ and } y \text{ variables} \\
  x - 8 &= -2y \quad \text{Isolate the } y \text{-term} \\
  \frac{x - 8}{-2} &= y \quad \text{Solve for the new } y \\
  \text{So, } f^{-1}(x) &= \frac{x - 8}{-2}
\end{align*}
\]

Note: Given the equation or “rule” of a one-to-one function, we can always swap the variables to find its inverse because, by definition, the inverse **reverses** the action of the original function.

In our previous example, we can clearly see how finding the inverse function is simply “undoing” the original function. In \( y = -2x + 8 \), we have basically two operations: first, we multiply the input by \(-2\), and then we add 8 to that product. The inverse function “undoes” the operations in the original function in reverse order. So, in \( f^{-1}(x) = \frac{x - 8}{-2} \), we first subtract the 8 from the input, and then divide by \(-2\). Consequently, it is commonly said that \( f^{-1} \) undoes whatever \( f \) does.
We can check some $x$-values and verify that $f^{-1}(x) = \frac{x - 8}{-2}$, in fact, undoes the actions of $y = -2x + 8$.

Let us evaluate $y = -2x + 8$ at $x = 1$: $-2(1) + 8 = 6$. So, we obtain the ordered pair $(1, 6)$.

Since $f^{-1}(x) = \frac{x - 8}{-2}$ is the inverse of the original function, then the reversed ordered pair must satisfy this inverse function.

Evaluating $f^{-1}(x)$ at $x = 6$, yields $\frac{6 - 8}{-2} = -2 = 1$. We can see how $f^{-1}$ has undone the operations in the original function.

Let us try with $x = -10$.

$y = -2x + 8$ at $x = -10 \rightarrow -2(-10) + 8 = 28$. We obtain $(-10, 28)$.

$f^{-1}(x) = \frac{x - 8}{-2}$ at $x = 28 \rightarrow \frac{28 - 8}{-2} = \frac{20}{-2} = -10$. The reversed pair is $(28, -10)$.

We have examined how $f^{-1}$ can be found by undoing the actions or operations in $f$ in reversed order. Let us now summarize the steps we can follow to find the inverse of a one-to-one function symbolically.

### Finding the Inverse of a Function

If $f$ is a one-to-one function defined by $y = f(x)$, perform the following steps to find its inverse:

1. Replace $f(x)$ with $y$.
2. Swap the input and the output (that is, interchange $x$ and $y$).
3. Solve the new equation for $y$.
4. Let $y = f^{-1}(x)$. That is, assign the name $f^{-1}(x)$ to the resulting inverse function.

Note: When applying step (3), if the new equation cannot be solved for $y$, then the original function has no inverse function.

### EXAMPLE 5

Find the inverse of each function, if it exists. If the inverse exists, state the domain and range for the original function and the inverse.

a. $f(x) = \frac{2x - 7}{5}$  
   b. $f(x) = x^4 - 3$  
   c. $f(x) = \sqrt[3]{x + 2}$

**Solution**

a. $f(x) = \frac{2x - 7}{5}$, which is equivalent to the linear function $f(x) = \frac{2}{5}x - \frac{7}{5}$, is one-to-one. Applying the steps to find the inverse of a function, we have

\[
\begin{align*}
f(x) &= \frac{2x - 7}{5} \\
y &= \frac{2x - 7}{5} \quad \text{Replace $f(x)$ with $y$} \\
x &= \frac{2y - 7}{5} \quad \text{Interchange the $x$ and $y$ variables} \\
5x + 7 &= 2y \quad \text{Isolate the $y$-term} \\
\frac{5x + 7}{2} &= y \quad \text{Solve for $y$}
\end{align*}
\]

So, $f^{-1}(x) = \frac{5x + 7}{2}$. 
The domain and range for \( f(x) = \frac{2x - 7}{5} \) is \((-\infty, \infty)\). The domain and range of \( f^{-1}(x) = \frac{5x + 7}{2} \) is \((-\infty, \infty)\).

**b.** In the function \( f(x) = x^4 - 3 \), we can see how two distinct \( x \)-values could yield the same \( y \)-value. For example, \( f(-1) = (-1)^4 - 3 = -2 \) and \( f(1) = (1)^4 - 3 = -2 \). Furthermore, when we examine the graph of the function, we clearly see that it does not pass the horizontal line test.

The function \( f(x) = x^4 - 3 \) is not one-to-one, thus it has no inverse function.

**c.** To find the inverse of \( f(x) = \sqrt[3]{x + 2} \), we apply the corresponding steps.

\[
\begin{align*}
  f(x) &= \sqrt[3]{x + 2} \\
  y &= \sqrt[3]{x + 2} \\
  x &= \sqrt[3]{y + 2} \\
  (x)^3 &= \left(\sqrt[3]{y + 2}\right)^3 \\
  x^3 &= y + 2 \\
  x^3 - 2 &= y
\end{align*}
\]

Hence, \( f^{-1}(x) = x^3 - 2 \). The domain and range for both functions is \((-\infty, \infty)\).

Let us now graph \( f(x) = \frac{2x - 7}{5} \) and its inverse, \( f^{-1}(x) = \frac{5x + 7}{2} \) on the same pair of axes, along with the equation \( y = x \). 
Recall that if the graph of the original function contains a point \((a, b)\), then the graph of the inverse function will contain the point \((b, a)\). The graph of a point \((b, a)\) is the reflection of the point \((a, b)\) across the line \(y = x\). Notice this property with the sample points \((6, 1)\), \((1, 6)\), and \((3.5, 0)\) and \((0, 3.5)\) on the graphs, respectively, as shown.

Since the graph of any point \((b, a)\) is the reflection of the point \((a, b)\) across the line \(y = x\), the graphs of the function and its inverse are reflections of each other about the line \(y = x\).

**The Graphs of Inverse Functions**

The graphs of a function and its inverse are symmetric about the line \(y = x\).

**EXAMPLE 6**

The graph of a function, \(f(x)\), is shown next. Graph \(f^{-1}(x)\) along with the line \(y = x\).

**Solution**

We can use symmetry about the line \(y = x\) to graph the inverse function. The points \((-2, -6)\), \((0, 2)\), \((1, 3)\), and \((2, 10)\) lie on the graph of \(f(x)\), therefore, the points \((-6, 2)\), \((2, 0)\), \((3, 1)\), and \((10, 2)\) will lie on the graph of \(f^{-1}(x)\). We can plot the points and connect them with a smooth curve to construct the graph of the inverse.
We can see that both graphs are symmetric about the line $y = x$.

**EXAMPLE 7**

The following graph shows the relationship between values on the Celsius and Fahrenheit scales. The function displayed, $f(x) = \frac{9}{5}x + 32$, converts a temperature of $x$ degrees Celsius to degrees Fahrenheit. Use the graph of $f(x)$ to answer the following:

a. Estimate and interpret $f(10)$ and $f(35)$. Plot the corresponding points on the graph.

b. Find the inverse of the given function.

c. Find and interpret $f^{-1}(32)$ and $f^{-1}(77)$.

d. Graph the inverse function on the same coordinate plane along with the line $y = x$, and use a point from each temperature function to confirm the symmetry.

**Solution**

a. From the graph, we can estimate that $f(10) = 50$ and $f(35) = 95$. This means that $10^\circ C$ is equivalent to $50^\circ F$ and $35^\circ C$ is the same as $95^\circ F$. 
b. We will find \( f^{-1}(x) \) by following the steps to find the inverse of a function symbolically.

\[
f(x) = \frac{9}{5}x + 32 \quad \rightarrow \quad y = \frac{9}{5}x + 32 \quad \rightarrow \quad x = \frac{9}{5}y + 32 \quad \rightarrow \quad \frac{5}{9}y = x - 32 \quad \rightarrow \quad y = \frac{5}{9}(x - 32)
\]

Therefore; \( f^{-1}(x) = \frac{5}{9}(x - 32) \) or, equivalently, \( f^{-1}(x) = \frac{5x - 160}{9} \).

c. \( f^{-1}(32) = \frac{5}{9}(32 - 32) = 0 \) and \( f^{-1}(77) = \frac{5}{9}(77 - 32) = 25 \). This means that 32°F is equivalent to 0°C and 77°F is the same as 25°C.

d. The graphs of \( f(x) \) and \( f^{-1}(x) \) are shown.

We can see that both graphs are symmetric about the line \( y = x \). The point (10, 50) lies on the original function, \( f(x) \), and it tells us that 10°C is equivalent to 50°F. On the other hand, the point (50, 10), which lies on the inverse function \( f^{-1}(x) \), denotes that 50°F is equivalent to 10°C.

**EXAMPLE 8**

Ken works in sales and makes a base salary of $42,000 a year plus a commission of 20% of the total sales.

a. Write a function that models Ken's total annual income, \( I \), in dollars, in terms of the total sales, \( x \), in dollars.

b. Use your grapher to graph this function. Include your selected window and label the axes in the context of the problem. Explain why this is a one-to-one function.
c. Find the inverse function.
d. Find and interpret \( I^{-1}(60000) \) in terms of the problem.

**Solution**

a. The function that models Ken’s total annual income in dollars is \( I = 42000 + 0.20x \), where \( x \geq 0 \).

b. \( I \): Income

\[ x: \text{Total sales} \]

\[ [0, 1000000, 100000] \text{ by } [0, 242000, 20000] \]

This is an increasing linear function, which passes the horizontal line test.

c. We calculate the inverse function, \( I^{-1}(x) \), as follows

\[
\begin{align*}
I &= 42000 + 0.20x \\
x &= 42000 + 0.20I \\
\frac{x - 42000}{0.20} &= I \\
5x - 210000 &= I \\
\text{So, } I^{-1}(x) &= 5x - 210000.
\end{align*}
\]

d. \( I^{-1}(60000) = 90000 \). This means that if Ken’s income was \( \$60,000 \), his total number of sales was \( \$90,000 \).

**Inverse Functions with Restricted Domains**

As we have learned, a function that is not one-to-one will not have an inverse function. In those cases, we can try to restrict the domain of the function such that there is a “limited” domain where the function would be one-to-one. Consider the function \( f(x) = x^2 - 2 \). Let us attempt to find its inverse function.

\[
f(x) = x^2 - 2 \rightarrow y = x^2 - 2 \rightarrow x = \pm \sqrt{y + 2}
\]

This cannot be the inverse function, since it is *not* a function! Observe that one input would have two outputs. For example, if \( x = 7 \), then \( y = \pm \sqrt{9} = \pm 3 \). Notice as well how the graph of the original function clearly displays that it did not satisfy the horizontal line test.
However, if we restrict the domain of \( f(x) = x^2 - 2 \) to, let’s say, \( x \geq 0 \), see how we create a “limited” domain, \([0, \infty)\), where the function is one-to-one. Observe from the graph that the range is \( y \geq -2 \) or \([-2, \infty)\).

Now we can see that the inverse function on the limited domain is \( y = \sqrt{x + 2} \) or \( f^{-1}(x) = \sqrt{x + 2} \). The graphs of the given function and its inverse are shown.

Notice that the domain of \( f^{-1}(x) \) is \( x \geq -2 \) or \([-2, \infty)\), and its range is \( y \geq 0 \) or \([0, \infty)\).

**EXAMPLE 9**

T. J. received a cruise vacation from his grandparents as a college graduation gift. As T. J. was enjoying the leisurely cruise and gazing at the ocean waves, he wondered how far it was to the horizon. The distance to the horizon can be calculated with the formula \( d = 1.169\sqrt{h} \), where \( d \) represents nautical miles and \( h \) is the height of eye, that is, the distance that the eyes are off the surface of the water, where \( h \geq 0 \). Use the horizon distance formula to answer the following questions.

**Little Fact:** A nautical mile (Nm) is an international unit of distance used in sea navigation. One Nm is about 1852 meters or approximately 6076 feet. Sources: www.boatsafe.com; www.ariesmarine.com; www.history.nasa.gov

a. If T. J. is 6 feet 3 inches tall, and the ship is 200 feet above sea level, calculate the distance to the horizon.

b. Find the inverse function.

c. Find and interpret \( f^{-1}(16.727) \).
5.1 Finding the Inverse: Numerical, Graphical, and Symbolic Approaches

Solution

a. The ship’s height is 200 feet and T.J.’s eyes (3 inches below his hairline) are 6 feet from the deck floor. So, T.J.’s height of eye is 206 feet above the surface of the water. Therefore,

\[ d = 1.169 \sqrt{206} = 16.778 \text{ Nm}. \]

b. To find the inverse function, we have

\[ d = 1.169 \sqrt{h} \]
\[ h = 1.169 \sqrt{d} \]
\[ \frac{h}{1.169} = \sqrt{d} \]
\[ \left( \frac{h}{1.169} \right)^2 = (\sqrt{d})^2 \]
\[ 0.7317h^2 = d. \]

So, \( f^{-1}(h) = 0.7317h^2 \) for \( h \geq 0. \)

c. \( f^{-1}(16.727) = 0.7317(16.727)^2 = 204.72 \). This means that if the distance to the horizon in nautical miles is 16.727, the person’s height of eye is 204.72 feet above the surface of the water.

EXAMPLE 10

The number of flash drives a new retail store is willing to supply for price, \( p \), dollars is given by \( p(d) = \frac{1}{2}d^2 + 15 \), where \( d \) represents the number of flash drives in thousands.

Little Facts: A flash drive is a small, ultra-portable data storage device which has a built-in USB (Universal Serial Bus). It is also known as jump drive, thumb drive, and keychain drive, amongst other names. The first flash drives could store only a few megabytes of data, but the modern ones are capable of storing several gigabytes of information. Dov Moran, an Israeli engineer, invented the flash drive in 1998. IBM was the first North American seller of a USB flash drive, and marketed an 8 MB version of the product in 2001 under the “Memory Key” nickname. Sources: www.techterms.com; www.usb-drive-flash.com

a. Decide if this function is one-to-one. Explain your decision.

b. Determine the restricted domain of this function in terms of the problem.

c. Find the inverse function in the restricted domain.

d. Calculate and interpret \( p^{-1}(29.95) \). Round your answer to the nearest whole number.

e. State the domain and range of \( p(d) \) and \( p^{-1}(d) \) in the context of the problem.

Solution

a. The function represents a concave up parabola and it does not pass the horizontal line test.

b. The number of flash drives produced cannot be negative, thus \( d \geq 0 \), or \([0, \infty)\).

c. The inverse in the restricted domain is shown as follows.

\[ p(d) = \frac{1}{2}d^2 + 15 \]
\[ y = \frac{1}{2}d^2 + 15 \]
\[ d = \frac{1}{2}y^2 + 15 \]
\[ 2d = y^2 + 30 \]
\[ 2d - 30 = y^2 \]
\[ \sqrt{2d - 30} = y \]
d. \( p^{-1}(29.95) = \sqrt{2(29.95)} - 30 = 5.4681 \). If the price per flash drive is $29.95, the retail store will supply 5,468 flash drives.

e. The domain and range of \( p(d) \) are \([0, \infty) \) and \([15, \infty) \), respectively. The domain and range of \( p^{-1}(d) \) are \([15, \infty) \) and \([0, \infty) \), respectively.

### 5.1 IN-CLASS PRACTICE

1. Determine whether each of the following functions is one-to-one. If the function is not one-to-one, explain why.
   - a. \( f(x) = -x \)
   - b. \( f(x) = (2x - 7)^2 - 4 \)
   - c. \( f(x) = -|x - 6| \)
   - d. \( f(r) = \sqrt{3r - 1}, \) for \( r \geq \frac{1}{3} \)

2. Find the inverse function for each of the following:
   - a. \( f(x) = 5x - 8 \)
   - b. \( f(x) = \frac{x - 2}{3} \)
   - c. \( f(x) = \frac{3}{2x + 1} \)

3. The points \((-11, -6)\) and \((1, -2)\) satisfy the function \( g(x) = \frac{x - 7}{3} \). Use this information to show that \( g(x) \) is the inverse function of \( f(x) = 3x + 7 \). Graph both functions with your grapher to confirm that they are symmetric about the line \( y = x \).

4. The graph of a function, \( f(x) \), is shown. Graph \( f^{-1}(x) \) along with the line \( y = x \).

5. As of June 18, 2011, $1.00 was equivalent to 0.6990 Euros.
   - a. Write a function \( f \) that represents the number of Euros in terms of the number of dollars, \( x \).
   - b. Find the inverse of your function.
   - c. Find and interpret \( f^{-1}(18) \). Round your answer to 2 decimal places.

### 5.1 Exercises

In exercises 1–6, determine whether each of the following functions is one-to-one. If the function is not one-to-one, explain why.

1. \( f(x) = -3 - x^2 \)
2. \( f(x) = 4 + \sqrt{3x} \)
3. \( f(x) = 4 + 3\sqrt{x}, \) \( x \geq 0 \)
4. \( f(x) = 2|x| \)
5. \( f = \{(−3, 17), (6, 19), (0, −5), (−2, −13)\} \)
6. \( f = \{(-1, 3.5), (5, 15.5), (−2, 5), (1, 3.5), (3, 7.5)\} \)
In exercises 7–10, determine whether each table represents a one-to-one function. If not, explain why.

7. 
<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>-5</td>
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</tr>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
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<td>3</td>
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8. 
<table>
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<th>Output</th>
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<td>5</td>
</tr>
<tr>
<td>8</td>
<td>-18</td>
</tr>
<tr>
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<td>12</td>
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9. 
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<td>-10</td>
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<tr>
<td>5</td>
<td>11</td>
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<tr>
<td>8</td>
<td>21</td>
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10. 
<table>
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<th>y</th>
</tr>
</thead>
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</tr>
<tr>
<td>0</td>
<td>8</td>
</tr>
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<td>6</td>
<td>20</td>
</tr>
<tr>
<td>9</td>
<td>53</td>
</tr>
</tbody>
</table>

In exercises 11–14, determine if the graphs represent functions that have an inverse; explain why or why not.

11. [Graph of a function]

12. [Graph of a function]

13. [Graph of a function]

14. [Graph of a function]

In exercises 15–18, state the domain and range of the inverse, given the following.

15. \( f = \{(-3, -15), (-1, -9), (0, -6), (5, 9)\} \)

16. \( f = \{(-11, 63), (-4, 28), (5, -17), (8, -32)\} \)

17. \( f(-4) = 6, f(0) = 8, \) and \( f(7) = 11.5 \)

18. \( f(-9) = -27, f(-6) = -8, \) and \( f(6) = 8 \)

In exercises 19–20, given the values do the following.

a. Make a table of values for \( f(x) \) and another table for its inverse, \( f^{-1}(x) \).

b. State the domain and range of each function.

19. \( f(-4) = 18, f(-1) = 28.46, \) and \( f(2) = 36 \)

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
x & f^{-1}(x) \\
\hline
\end{array}
\]
20. \( f(-20) = -75, \quad f(-3) = -24, \quad \text{and} \quad f(12) = 21 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f^{-1}(x) )</th>
</tr>
</thead>
</table>

In exercises 21–36, find the inverse function for each of the following.

21. \( f(x) = 2x + 6 \)
22. \( f(x) = 7x - 4 \)
23. \( f(x) = -3x - 11 \)

24. \( f(x) = \frac{x + 9}{3} \)
25. \( f(x) = \frac{10x - 3}{5} \)
26. \( f(x) = -\frac{16x - 3}{14} \)

27. \( f(x) = -\frac{8x + 5}{10} \)
28. \( f(x) = -\frac{25 - 9x}{15} \)
29. \( f(x) = \frac{7}{4x + 3} \)

30. \( f(x) = \frac{-5}{3x - 2} \)
31. \( f(x) = \frac{1}{2}x + 5 \)
32. \( f(x) = \frac{2}{3}x - 5 \)

33. \( f(x) = -\frac{3}{5}x - \frac{3}{4} \)
34. \( f(x) = -\frac{7}{2} - \frac{6}{7}x \)
35. \( f(x) = \frac{x + 2}{x - 3} \)

In exercises 37–45, the graph of a function \( f(x) \), is shown; graph \( f(x) \) and \( f^{-1}(x) \) along with the line \( y = x \).
In exercises 46–48 the graph of a function \( f(x) \) is given. Graph its inverse \( f^{-1}(x) \) using the 2 points shown as a guide. Re-label the points and the tick marks accordingly.

46.

47.

48.

In exercises 49–52, find the inverse of each function, if it exists. If the inverse exists, state the domain and range for the original function and the inverse. Explain why the inverse does or does not exist.

49. \( f(x) = (x + 2)^2 - 5 \)

50. \( f(x) = \sqrt{x - 3} \)

51. \( f(x) = \sqrt[3]{x - 6} \)

52. \( f(x) = \sqrt{x + 1} \)

53. Hiroki bought a new boat last month after receiving a large bonus from his employer. He paid title fees and also paid 7% of the purchase price for taxes. The equation for the total cost of the boat, \( f(x) \), is given by \( f(x) = 17150 + 0.07x \), where \( x \) is the purchase price.

a. Find \( f(16500) \) and interpret its meaning.

b. Find the inverse function.

c. Find \( f^{-1}(18305) \) and interpret its meaning.

d. Using your answer from part (a), determine how much Hiroko paid in taxes and how much he paid in title fees.

54. Bruce is a race car enthusiast and decided to purchase his classic dream car, a 1969 SS Chevy Chevelle. Bruce plans to compete in the “Pinks All Out” at the zMAX Dragway in Concord, North Carolina, the only four lane dragway in the United States. Bruce found the car at a local dealer who only sells classic cars. He paid dealer, tax, and title fees, and then paid 5.75% of the purchase price for sales tax. The equation for the total cost of the car, \( f(x) \), is given by \( f(x) = 30500 + 0.0575x \), where \( x \) is the purchase price.

Little Facts: Pinks All Out is a show on Speed TV in which people race their cars on a dragway for prize money. The show is a spinoff from “Pinks,” in which the losing contestant received a “pink slip”; their motto was “Lose the race, lose your ride.” In April 2010, contestants competed in Pinks All Out at the zMAX dragway on a four-lane 3,835 feet dragway. The seating capacity at the track is approximately 30,000. Sources: www.speedtv.com; www.charlottemotorspeedway.com

a. Find \( f(28000) \) and interpret its meaning.

b. Find the inverse function.

c. Find \( f^{-1}(32110) \) and interpret its meaning.

d. Using your answer from part (a), determine how much he paid for the dealer, tax, and title fees, and also for sales tax.
55. Jordan purchased a new 2011 Chevy Volt after trading in his suburban. He paid dealer and title fees and also paid 6.5% of the purchase price for taxes. The equation for the total cost of the car, \( f(x) \) is given by \( f(x) = 36210 + 0.065x \), where \( x \) is the purchase price.

**Little Facts:** With a MSRP of around $40,000, the 150 horsepower 2011 electric Chevy Volt debuted in November, 2010. The full-charge cycle is around eight hours when plugged into an electrical outlet and will run for up to forty miles.

Source: www.chevrolet.com

a. Find \( f(34500) \) and interpret its meaning. Round answer to 2 decimal places.
b. Find the inverse function. Round dollar amount to 2 decimal places.
c. Find \( f^{-1}(38452.50) \) and interpret its meaning.
d. Using your answer from part (a), determine how much Jordan paid in taxes and how much he paid in dealer and title fees. Round answers to 2 decimal places.

56. Miguel travels frequently due to his management position at work and has noticed that gas prices continue to increase. To save money, he trades in his V-8 pickup truck for a 4-cylinder compact car. The total loan (interest included) is $24,222 over a 66 month period. His monthly payment will be $367, with the outstanding amount denoted by \( f(x) = 24222 - 367x \), where \( x \) is the number of months paid towards the loan.

a. Find \( f(60) \) and interpret its meaning.
b. Find the inverse of the given function.
c. Find \( f^{-1}(11377) \) and interpret its meaning.

57. An object is dropped from a height of 80 feet. Its height in feet, \( h \), after \( t \) seconds is modeled by the function \( h(t) = -16t^2 + 80 \) for \( t \geq 0 \).

a. Find \( h(1.5) \) and interpret your answer.
b. Find the inverse of the given function.
c. Find \( h^{-1}(50) \) to 2 decimal places and interpret its meaning.

58. The profit of a company, in thousands, can be approximated by \( P(x) = \sqrt{22.3x + 4.8} \), where \( x \) denotes time in months for \( x \geq 0 \).

a. Find \( P(10) \) to 1 decimal place and interpret your answer.
b. Find the inverse of the given function.
c. Find \( P^{-1}(15.1) \) to the nearest whole number and interpret its meaning.
d. Find \( P^{-1}(25) \) to the nearest whole number and interpret its meaning.

59. An employee of a sales company receives a salary of $30,000 plus 2.5% commission on sales. The function \( S(x) = 30000 + 0.025x \) describes the employee’s salary, where \( x \) represents the amount of sales in dollars.

a. Complete the table of values for \( S(x) \):

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<thead>
<tr>
<th>( x )</th>
<th>20,000</th>
<th>80,000</th>
<th>100,000</th>
<th>800,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Find \( S^{-1}(x) \) symbolically.
c. Complete the table of values for \( S^{-1}(x) \):

<table>
<thead>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

d. Find and interpret \( S(80000) \).
e. Find and interpret \( S^{-1}(60000) \).

60. An employee of a marketing company receives a salary of $45,000 plus 3.5% on sales. The function \( S(x) = 45000 + 0.035x \) describes the employee’s salary, where \( x \) represents the amount of sales in dollars.

a. Complete the table of values for \( S(x) \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>15,000</th>
<th>35,000</th>
<th>50,000</th>
<th>75,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
b. Find $S^{-1}(x)$ symbolically.

c. Complete the table of values for $S^{-1}(x)$: Round values to the nearest whole number.

<table>
<thead>
<tr>
<th>x</th>
<th>$S^{-1}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>46,225</td>
<td>47,625</td>
</tr>
<tr>
<td>48,000</td>
<td>50,000</td>
</tr>
</tbody>
</table>

d. Find and interpret $S(35000)$.

e. Find and interpret $S^{-1}(57000)$.

61. A group of Girl Scouts plans to make small fruit baskets to raise money for the elderly in an assisted living home. Their fixed cost for the baskets, fruit, and ribbon is $125, and the cost to have a florist make them is $5.50 per basket.

**Little Facts:** On March 12, 1912 Juliette Gordon Low registered eighteen girls to form the first troop of American Girl Guides; one year later the name was changed to Girl Scouts. Nicknamed “Daisy,” Juliette was born on October 31, 1860 in Savannah, Georgia. Before her marriage to William Low, Juliette suffered from chronic ear infections which left her near deaf in one ear. She lost all hearing in her other ear when on her wedding day a piece of rice thrown for good luck became lodged in her ear. The Girl Scouts encourage young women to become professionals in the arts, science, and business areas and welcome girls from all walks of life and those with disabilities. Source: www.girlscouts.org

a. Write a function that models the Girl Scout’s cost function $C(x)$, in terms of the number of $x$ baskets sold.

b. Find $C(95)$ and interpret its meaning.

c. Use your grapher to graph this function. Include your selected window and label the axes in the context of the problem. Explain why this is a one-to-one function.

d. Find the inverse function.

e. Find and interpret $C^{-1}(500)$ in terms of the problem. Round your answer to the nearest whole number.

62. A soccer team plans to make homemade bracelets to raise money for team shirts. A parent of one of the players determines that their fixed cost is $30 for the beads and their cost to make each bracelet is $10.

a. Write a function that models the soccer team’s cost function $C(x)$, in terms of the number of $x$ bracelets sold.

b. Find $C(80)$ and interpret its meaning.

c. Use your grapher to graph this function. Include your selected window, and label the axes in the context of the problem. Explain why this is a one-to-one function.

d. Find the inverse function.

e. Find and interpret $C^{-1}(1800)$ in terms of the problem.

**Reflective Thought**

63. Analyze the following figure. Could these graphs represent inverse functions of each other? Use a complete sentence to explain why or why not.

64. Janet prefers to conduct an Internet search for a business phone number instead of using the regular phone book delivered by her telephone service provider. She types the name and location of the business and instantly receives the telephone number. Assume that the business has only one phone number assigned. If searching for the phone number via the Internet represents a function, state the inverse of this function, decide whether this inverse would also be a function, and explain your answer.
65. Which of the following represents a table of values for the inverse of \( f(x) = 3x + 5 \)? Explain your decision.

\[ \begin{array}{c|c|c|c|c}
   x  & f(x) & x  & f^{-1}(x) \\
   \hline
   -4 & 18 & 18 & -4 \\
   -1 & 28.46 & 28.46 & -1 \\
   2  & 36  & 36  & 2 \\
\end{array} \]

66. Is the following statement true or false? Explain your decision.

If \( f(x) = \frac{3x - 4}{10} \) then \( f^{-1}(x) = \frac{10}{3x - 4} \)

---

**Answer Key 5.1 Exercises**

1. Not one-to-one; this represents a parabola, which fails the horizontal line test.

3. One-to-one

5. One-to-one

7. One-to-one

9. Not one-to-one; two different inputs produce the same output.

11. Yes, passes the horizontal line test.

13. No, does not pass the horizontal line test.

15. \( D = \{-15, -9, -6, 9\}, R = \{-3, -1, 0, 5\} \)

17. \( D = \{6, 8, 11.5\}, R = \{-4, 0, 7\} \)

19. a. \( \begin{array}{c|c|c|c|c}
   x  & f(x) & x  & f^{-1}(x) \\
   \hline
   -4 & 18 & 18 & -4 \\
   -1 & 28.46 & 28.46 & -1 \\
   2  & 36  & 36  & 2 \\
\end{array} \)

b. \( f(x): D = \{-4, -1, 2\}, R = \{18, 28.46, 36\} \) \( f^{-1}(x): D = \{18, 28.46, 36\}, R = \{-4, -1, 2\} \)

21. \( f^{-1}(x) = \frac{1}{2}x - 3 \)

23. \( f^{-1}(x) = \frac{1}{3}x - \frac{11}{3} \)

25. \( f^{-1}(x) = \frac{1}{2}x + \frac{3}{10} \)

27. \( f^{-1}(x) = -\frac{5}{3}x - \frac{5}{8} \)

29. \( f^{-1}(x) = -\frac{3}{4}x - \frac{3}{4} \)

31. \( f^{-1}(x) = 2x - 10 \)

33. \( f^{-1}(x) = -\frac{5}{3}x - \frac{5}{4} \)

35. \( f^{-1}(x) = \frac{3}{x + 2} \)

37. 

39. 

41.
49. Answers may vary. An inverse does not exist since \( f(x) \) represents a parabola which fails the horizontal line test.

51. \( f^{-1}(x) = x^3 + 6 \); The inverse exists and the graph of \( f(x) \) and \( f^{-1}(x) \) pass the horizontal line test. The domain and range for both functions is \((-\infty, \infty)\).

53. a. 18305; if the purchase price was $16,500, the total cost was $18,305.
   b. \( f^{-1}(x) = \frac{x - 17,150}{0.07} \) or \( f^{-1}(x) = \frac{x}{0.07} - 245,000 \)
   c. 16500; if the total cost was $18,305, the purchase price was $16,500.
   d. He paid $1155 in taxes and $650 for title fees.

55. a. 38452.50; if the purchase price was $34,500, the total cost was $38,452.50.
   b. \( f^{-1}(x) = \frac{x - 36,210}{0.065} \) or \( f^{-1}(x) = \frac{x}{0.065} - 557,076.92 \)
   c. 34500; if the total cost was $38,452.50, the purchase price was $34,500.
   d. He paid $2242.50 in taxes and $1710 for dealer and title fees.

57. a. 44; in 1.5 seconds the height of the object is 44 feet above ground.
   b. \( h(t) = \sqrt{5 - \frac{t}{4}} \) or \( \sqrt{80 - \frac{t}{16}} \)
   c. 1.37; the height of the object is 50 feet above ground in 1.37 seconds.

59. a. \[
\begin{array}{c|cccc}
   x & 20,000 & 80,000 & 100,000 & 800,000 \\
   S(x) & 30,500 & 32,000 & 32,500 & 50,000 \\
\end{array}
\]
   b. \( S^{-1}(x) = \frac{x - 30,000}{0.025} \)
   c. \[
\begin{array}{c|ccccc}
   x & 30,500 & 32,500 & 33,000 & 60,000 \\
   S^{-1}(x) & 20,000 & 100,000 & 120,000 & 1,200,000 \\
\end{array}
\]
   d. When the sales amount is $80,000, the salary is $32,000.
   e. If the salary is $60,000, the sales amount is $1,200,000.

61. a. \( C(x) = 125 + 5.50x \), where \( x \geq 0 \).
   b. 647.50; the cost of selling 95 baskets is $647.50.
   c. Answers may vary. This is an increasing linear function, which passes the horizontal line test. Window used: \([0, 105, 15] \) by \([0, 720, 60] \)
EXAMPLE 1

Let \( f(x) = 3x^2 + 1 \) and \( g(x) = x - 6 \). Find each function and state its domain.

a. \((f + g)(x)\)  

Solution

a. \((f + g)(x) = f(x) + g(x) = (3x^2 + 1) + (x - 6) = 3x^2 + x - 5\)

The domain of the new function is the set of real numbers \( x \) contained in the domains of both \( f \) and \( g \). The domain for both \( f \) and \( g \) is the set of all real numbers, or \((-\infty, \infty)\). Therefore, the domain of \( f + g \) is \((-\infty, \infty)\).

b. \((f - g)(x) = f(x) - g(x) = (3x^2 + 1) - (x - 6) = 3x^2 + 1 - x + 6 = 3x^2 - x + 7\)

The domain of \( f - g \) is \((-\infty, \infty)\).

63. No; The two graphs are not symmetric about the line \( y = x \). Answers may vary.

65. Table c, the inputs and outputs of the original function have been interchanged. Answers may vary.
c. \((f \cdot g)(x) = f(x) \cdot g(x) = (3x^2 + 1)(x - 6) = 3x^3 - 18x^2 + x - 6\)

The domain of \(f \cdot g\) is \((-\infty, \infty)\).

d. \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{3x^2 + 1}{x - 6}\)

In the quotient function we have the restriction that \(g(x) \neq 0\). Therefore, the domain of \(\frac{f}{g}\) is the set of all real numbers, with the exception \(x \neq 6\), or \((-\infty, 6) \cup (6, \infty)\).

EXAMPLE 2

A computer video games company plans to develop a new game based on aliens and humans collaborating to inhabit a newly discovered planet with plenty of natural resources suitable for both species. The new game is expected to be sold at a high price. The projected revenue in dollars for this game for \(x\) units to be sold is modeled by \(R(x) = x(985 - 0.001x)\). The total cost of producing this game involves a fixed cost of $7,500,000 plus a variable cost of $45 per unit to be sold.

Little Facts: More than 60% of families in the United States play computer video games. The average cost to make a computer video game is about $15 million, but some games can cost more than $50 million to produce. As of 2010, Grand Theft Auto 4 and Starcraft II were the most expensive video games ever made, with a cost of over $100 million. A large group of video game developers project that after 2014, this production will be three times the size of the music industry, but some analysts project a decline due to the increasing development and production costs. Sources: www.seekingalpha.com; www.joystick.com; www.gamepro.com; www.digitalbattle.com; www.geek.com

a. Write the total cost function \(C(x)\), for this problem.
b. Find \(R(x) - C(x)\) and explain what it represents.
c. Find and interpret \(P(100000)\).

Solution

a. The total cost function is the sum of the variable cost and the fixed cost. Therefore, the total cost function is represented by \(C(x) = 45x + 7500000\).

b. \(R(x) - C(x)\) is the difference of the revenue function and the cost function, and it gives us the profit function, \(P(x)\). To find the difference function we first simplify the revenue equation.

\[R(x) = x(985 - 0.001x) \Rightarrow R(x) = 985x - 0.001x^2\]

The profit function is then given by

\[P(x) = (985x - 0.001x^2) - (45x + 7500000) = -0.001x^2 + 940x - 7500000.\]

c. \(P(100000) = -0.001(100000)^2 + 940(100000) - 7500000 = 76500000.\) If the computer video games company sells 100,000 units, they can expect a profit of $76,500,000.

EXAMPLE 3

Let \(f(x) = x^2 + 1\) and \(g(x) = 3x^2 - 5\). Evaluate each of the following.

a. \((f + g)(-2)\)  b. \((f - g)(a)\)  c. \((f \cdot g)(0.5)\)  d. \(\left(\frac{f}{g}\right)(0)\)

Solution

a. The sum function is \(f(x) + g(x) = (x^2 + 1) + (3x^2 - 5) = 4x^2 - 4\).

So, \((f + g)(-2) = 4(-2)^2 - 4 = 4(4) - 4 = 12\)

We can also evaluate each given function independently, and then add the resulting values.

\(f(-2) = (-2)^2 + 1 = 5\) and \(g(-2) = 3(-2)^2 - 5 = 7\)

So, \(f(-2) + g(-2) = 5 + 7 = 12\)
b. \((f - g)(a) = f(a) - g(a) = (a^2 + 1) - (3a^2 - 5) = a^2 + 1 - 3a^2 + 5 = -2a^2 + 6\)

c. \((f \cdot g)(0.5) = f(0.5) \cdot g(0.5) = (0.5^2 + 1) \cdot [3(0.5)^2 - 5] = (1.25)(-4.25) = -5.3125\)

If we decide to first find the product of the given functions and then evaluate, we get the same answer:
\[ f(x) \cdot g(x) = (x^2 + 1) \cdot (3x^2 - 5) = 3x^4 - 2x^2 - 5 \]

Thus, \((f \cdot g)(0.5) = 3(0.5)^4 - 2(0.5)^2 - 5 = -5.3125\)

d. \[ \left(\frac{f}{g}\right)(0) = \frac{0^2 + 1}{3(0)^2 - 5} = \frac{1}{-5} = -\frac{1}{5} \]

---

**EXAMPLE 4**

Use the graph to evaluate each function. State any undefined expression.

\(a. \ (f + g)(4) \quad b. \ (f \cdot g)(0) \quad c. \ (f/g)(1)\)

\[ \text{To find } (f + g)(4) \text{ we only need to add the corresponding } y \text{-values for } x = 4 \text{ on each individual function.} \]

\[ f(4) = 2 \quad \text{and} \quad g(4) = 9. \text{ Therefore, } (f + g)(4) = 2 + 9 = 11 \]

\[ b. \ (f \cdot g)(0) = f(0) \cdot g(0) = (0)(-3) = 0 \]

\[ c. \ (f/g)(1) = \frac{f(1)}{g(1)} \]

This expression is undefined since \(g(1) = 0\), and \(g(x)\) cannot equal 0 (recall that division by zero is undefined).

---

**EXAMPLE 5**

The following graph reflects the number of completers for degree (AA, AS, and AAS) and certificate programs at Valencia College, in Orlando FL, for the academic years 2003–2004 through 2008–2009. *AA = Associate in Arts, AS = Associate in Science, AAS = Associate in Applied Science*

Little Facts: Valencia College is considered an innovative leader in higher education with a national reputation for teaching excellence. It has four major campuses and two centers in Orlando, FL. In 1967, it began as Valencia Junior College. In 1971 its name changed to Valencia Community College until July 2011, when it officially became Valencia College. Source: www.valenciacollege.edu

\(a. \) Estimate \(d(2004 - 2005)\) and \(c(2004 - 2005)\).

\(b. \) Find and interpret \((d + c)(2004 - 2005)\).

\(c. \) Compare your answer to part (b) with the data displayed on the graph.

\(d. \) Estimate and interpret \((d - c)(2008 - 2009)\).

\(e. \) What does this model project in terms of the number of degree and certificate programs awarded at Valencia College.
Solution


c. According to the graph, the total number of degree and certificate programs awarded for the academic year 2004–2005 is approximately 7,300, which is the same value obtained for \((d + c)(2004 - 2005)\).


e. Should this model remain unaltered, the number of degree and certificate programs awarded at this institution will continue increasing during the coming years.

Composition of Functions

We have already performed arithmetic combinations of functions. Another way of combining functions is to form a composition of one function with another. Basically, in the composition of two functions we use the output of one function as the input of another function. This process is frequently referred to as “a function of a function.”

Let us explain this in more detail. Suppose we have \(f(x) = x^2 - 2\) and \(g(x) = 3x\). We will combine these functions by the composition of \(f\) and \(g\). To do this, we simply find \(f\) of \(g\), that is \(f(g(x))\). Notice that \(f(g(x))\) means that we are replacing the \(x\)-value in \(f\) with the function \(g(x)\), that is, we are using the function \(g\) as the input in \(f\). See the sequence of actions as follows.

\[
f(g(x)) \quad = \quad f(3x) \quad = \quad (3x)^2 - 2 \quad = \quad 9x^2 - 2
\]

function \(g(x)\) becomes the input for function \(f(x)\)
replace \(g(x)\) with \(3x\)
replace \(x\) in \(f(x) = x^2 - 2\)

What would \(g(f(x))\) represent? Now we are using the function \(f\) as the input in \(g\). See the sequence of actions as follows.

\[
g(f(x)) \quad = \quad g(x^2 - 2) \quad = \quad 3(x^2 - 2) \quad = \quad 3x^2 - 6
\]

function \(f(x)\) becomes the input for function \(g(x)\)
replace \(f(x)\) with \(x^2 - 2\)
replace \(x\) in \(g(x) = 3x\) with \(x^2 - 2\)

Observe that one function is “composed” of another function, thus, the new function is called a composite function. We can also call \(f(g(x))\) the composition of \(f\) and \(g\).
Let $f$ and $g$ be two functions.

- The composite function $f$ of $g$, denoted $f \circ g$, is defined as $(f \circ g)(x) = f(g(x))$ provided that $x$ is in the domain of $g$ and $g(x)$ is in the domain of $f$.

- The composite function $g$ of $f$, denoted $g \circ f$, is defined as $(g \circ f)(x) = g(f(x))$ provided that $x$ is in the domain of $f$ and $f(x)$ is in the domain of $g$.

$f \circ g$ can be read as: $g$ composed with $f$ or $g$ composed with $f$ or “the composition of $f$ and $g$” or “the composition of $g$ and $f$” or “the composite of $f$ with $g$” or “the composite of $g$ with $f$”

Note: For most functions, composition is not commutative, that is, $f \circ g \neq g \circ f$.

### CAUTION
Do not confuse $f \circ g$ with $f \cdot g$. The latter is the product of two functions. Notice that the composite function has an open circle, and not the multiplication dot.

The following diagram illustrates the composition of $f \circ g$. Observe that $x$ is the input for the function $g$, with output $g(x)$. In the composite, now $g(x)$ becomes the input for the function $f$.

### CAUTION
Since $f \circ g \neq g \circ f$ for most functions, the order in which we apply each function does make a difference. A good hint to follow is to remember that the first function that appears in the composition is, in fact, the last function to apply. For example, in $f \circ g = f(g(x))$, first we apply $g$, then $f$.

### EXAMPLE 6
Given $f(x) = 2x - 7$, $g(x) = x^2 + 3$, and $h(x) = \frac{1}{x - 2}$, find each of the following. State the domain of each composite function.

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

c. $(h \circ f)(x)$

d. $(f \circ h)(x)$

#### Solution

a. $(f \circ g)(x)$ equals $f(g(x))$, thus the function $g(x)$ will be the input for $f(x)$. So, we only need to replace $g(x)$ with $x^2 + 3$, and then simplify:

$(f \circ g)(x) = f(g(x)) = f(x^2 + 3)$.

Since $f(x) = 2x - 7$, we have

$f(x^2 + 3) = 2(x^2 + 3) - 7 = 2x^2 + 6 - 7 = 2x^2 - 1$.

Therefore, $(f \circ g)(x) = 2x^2 - 1$.

Observe that there are no restrictions in the domain of the newly formed composite function. The domain is the set of all real numbers, or $(-\infty, \infty)$.

b. $(g \circ f)(x) = g(f(x))$, and in this case the function $f(x)$ will be the input for $g(x)$.

$(g \circ f)(x) = g(f(x)) = g(2x - 7)$
Since \( g(x) = x^2 + 3 \), then
\[
g(2x - 7) = (2x - 7)^2 + 3 = 4x^2 - 28x + 49 + 3 = 4x^2 - 28x + 52.
\]
Hence, \( (g \circ f)(x) = 4x^2 - 28x + 52 \), and its domain is \((-\infty, \infty)\).

Notice that in \( f(g(x)) \) we substituted the expression for \( g(x) \) into \( f(x) \), \( g \) was the input for \( f \). On the other hand, in \( g(f(x)) \) we substituted the expression for \( f(x) \) into \( g(x) \), \( f \) became the input for \( g \).

c. \( (h \circ f)(x) = h(f(x)) \), so \( f \) is the input for \( h \). Replacing \( f(x) \) with \( 2x - 7 \) and simplifying, we have
\[
(h \circ f)(x) = h(2x - 7).
\]
Since \( h(x) = \frac{1}{x - 2} \), then
\[
h(2x - 7) = \frac{1}{(2x - 7) - 2} = \frac{1}{2x - 9}.
\]
Therefore, \( (h \circ f)(x) = \frac{1}{2x - 9} \). Since replacing \( x \) with \( \frac{9}{2} \) would yield a zero in the denominator, the domain of the composite function is the set of all real numbers, with the exception \( x \neq \frac{9}{2} \), or \((-\infty, \frac{9}{2}) \cup \left(\frac{9}{2}, \infty\right)\).

d. \( (f \circ h)(x) = f(h(x)) \), which means that \( h \) is the input for \( f \). Since \( h(x) = \frac{1}{x - 2} \), then \( x = 2 \) will not be in the domain of \( h \), nor the domain of the composite function.
\[
(f \circ h)(x) = f(h(x)) = f\left(\frac{1}{x - 2}\right) = 2\left(\frac{1}{x - 2}\right) - 7 = \frac{2}{x - 2} - 7.
\]
The domain is \((-\infty, 2) \cup (2, \infty)\), or the set of all real numbers with the exception \( x \neq 2 \).

---

**EXAMPLE 7**

Given \( f(x) = x^2 - 8 \) and \( g(x) = 2 - x \) find the following.

a. \( (f \circ g)(3) \)  

b. \( (f \circ g)(0) \)  

c. \( (g \circ f)(-5) \)

**Solution**

a. We can find \( (f \circ g)(3) \) following any of these approaches.

1. We can calculate the composite function in terms of \( x \), and then evaluate \( (f \circ g)(x) \) when \( x = 3 \).
\[
(f \circ g)(x) = f(g(x)) = f(2 - x) = (2 - x)^2 - 8 = x^2 - 4x + 4 - 8 = x^2 - 4x - 4.
\]
So, \( (f \circ g)(3) = (3)^2 - 4(3) - 4 = -7 \).

2. Another way is to first calculate the value of the inside function “\( g \)” when \( x = 3 \):
\[
g(3) = 2 - 3 = -1.
\]
Now substitute the resulting value into \( f \) and evaluate:
\[
f(-1) = (-1)^2 - 8 = 1 - 8 = -7.
\]
Observe that both methods will give us the same solution.

b. We will follow the second approach to find \( (f \circ g)(0) \).
\[
g(0) = 2 - 0 = 2 \quad \text{and} \quad f(2) = (-2)^2 - 8 = 4 - 8 = -4
\]
Hence, \( (f \circ g)(0) = -4 \).
c. Let us find \((g \circ f)(-5)\) by applying the second method. We will verify our answer by first finding the composite function in terms of \(x\), and then evaluating \((f \circ g)(x)\) when \(x = -5\).

\[
f(-5) = (-5)^2 - 8 = 17 \quad \text{and} \quad g(17) = 2 - 17 = -15
\]

So, \((g \circ f)(-5) = -15\).

Finding the composite function in terms of \(x\) and then evaluating \((f \circ g)(x)\) when \(x = -5\), we have:

\[
(g \circ f)(x) = g(f(x)) = g(x^2 - 8) = 2 - (x^2 - 8) = 2 - x^2 + 8 = -x^2 + 10.
\]

Therefore, \((g \circ f)(-5) = -(5)^2 + 10 = -25 + 10 = -15\). Again, both methods will provide the same answer.

**EXAMPLE 8**

Given \(f(x) = \sqrt{3x + 1}\), \(g(x) = 4x - 9\), and \(h(x) = x^2 + 4\), find \((f \circ g \circ h)\left(\frac{1}{2}\right)\).

**Solution**

\((f \circ g \circ h)\left(\frac{1}{2}\right)\) implies that we must first find \(g \circ h\) and then compose the function \(f\) with this result.

Let us find \((g \circ h)\left(\frac{1}{2}\right)\).

\[
h\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + 4 = \frac{1}{4} + \frac{16}{4} = \frac{17}{4} \quad \text{and} \quad g\left(\frac{17}{4}\right) = 4\left(\frac{17}{4}\right) - 9 = 17 - 9 = 8
\]

Thus the result of this first composition \((g \circ h)\left(\frac{1}{2}\right) = 8\).

Now we find \(f(8) = \sqrt{3(8) + 1} = \sqrt{25} = 5\). Hence, \((f \circ g \circ h)\left(\frac{1}{2}\right) = 5\).

**EXAMPLE 9**

A company that specializes in electrical appliances is offering a winter special on their French-door refrigerators. The suggested retail price of a French-door is \(p\) dollars. The company is offering a rebate of $150 and a 15% discount on new French-door refrigerators.

a. Write a function, \(R(p)\), that represents the cost of a French-door refrigerator after the dealer’s rebate.

b. Write a function, \(D(p)\), that represents the cost of a French-door refrigerator after the 15% discount.

c. Find and interpret \((R \circ D)(p)\).

d. Find and interpret \((R \circ D)(1300)\).

e. Find and interpret \((D \circ R)(p)\).

f. Find and interpret \((D \circ R)(1300)\).

g. Which composite function is the better option for the consumer, \(R \circ D\) or \(D \circ R\)? Explain your decision.

**Solution**

a. The cost of the refrigerator after the dealer’s rebate is found by subtracting the rebate from the retail price.

\[
R(p) = p - 150
\]

b. Since there is a 15% discount, the customer will only pay 85% of the retail price.

\[
D(p) = 0.85p
\]
c. \((R \circ D)(p) = R(D(p)) = R(0.85p) = 0.85p - 150\). This is the price after the discount followed by the rebate.

d. \((R \circ D)(1300) = 0.85(1300) - 150 = 955\) dollars. The price of a $1,300 French-door refrigerator discounted at 15% then followed by a $150 rebate will be $955.

e. \((D \circ R)(p) = D(R(p)) = D(p - 150) = 0.85(p - 150)\). This is the price after the rebate followed by the discount.

f. \((D \circ R)(1300) = 0.85(1300) - 127.50 = 977.50\) dollars. The price of a $1,300 French-door refrigerator minus the $150 rebate then discounted at 15% will be $977.50.

g. \(R \circ D\) is the better offer. Taking the 15% discount off the regular price first results in a greater price reduction.

---

**EXAMPLE 10**

Given the following graph, find the following.

a. \((f \circ g)(0)\)  
b. \((g \circ f)(0)\)  
c. \((g \circ g)(-2)\)

**Solution**

a. First we find \(g(0) = 2\). Now we evaluate \(f(2) = -10\). So, \((f \circ g)(0) = -10\).

b. \((g \circ f)(0) = g(f(0)) = g(0) = g(-8) = -22\)

c. Observe that here we are composing “g” with itself.
\[(g \circ g)(-2) = g(g(x)) = g(g(-2)) = g(-4) = -10\]

---

**EXAMPLE 11**

Use the tables of \(f(x)\) and \(g(x)\) to complete the tables of values for \((f \circ g)(x)\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-1)</th>
<th>(3)</th>
<th>(5)</th>
<th>(7)</th>
<th>(x)</th>
<th>(-2)</th>
<th>(4)</th>
<th>(6)</th>
<th>(10)</th>
<th>(x)</th>
<th>(-2)</th>
<th>(4)</th>
<th>(6)</th>
<th>(10)</th>
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</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>0</td>
<td>-5</td>
<td>3</td>
<td>9</td>
<td>(g(x))</td>
<td>3</td>
<td>7</td>
<td>-1</td>
<td>3</td>
<td>((f \circ g)(x))</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>

**Solution**

Since \((f \circ g)(x) = f(g(x))\), we will first find \(g(x)\) for each given \(x\) and then evaluate \(f(x)\) with the corresponding output of \(g(x)\), like this:

\[f(g(-2)) = f(3) = -5\], so, \((f \circ g)(-2) = -5\]  
\[f(g(4)) = f(7) = 9\], so, \((f \circ g)(4) = 9\]

\[f(g(6)) = f(-1) = 0\], so, \((f \circ g)(6) = 0\]  
\[f(g(10)) = f(3) = -5\], so, \((f \circ g)(10) = -5\]
The complete table is shown.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$4$</th>
<th>$6$</th>
<th>$10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(f \circ g)(x)$</td>
<td>$-5$</td>
<td>$9$</td>
<td>$0$</td>
<td>$-5$</td>
</tr>
</tbody>
</table>

**Composite Functions with the Graphing Calculator**

We can use the graphing calculator to explore compositions of functions.

---

**EXAMPLE 12**

Given $f(x) = x^2 + 3x + 1$ and $g(x) = 4x + 3$, find $(f \circ g)(-4)$ with the grapher. Confirm your answer algebraically.

**Solution**

First we enter functions $f$ and $g$ into Y1 and Y2 respectively. Under Y3, we will enter the expression for the composite function $f \circ g$ as Y1(Y2).

To enter Y1(Y2), once we have the cursor under the Y3, we press the key, select Y-VARS and choose option 1 (Function). From the list of functions, we then select the sought after functions Y1 and Y2. See the corresponding screens.

Since we are interested in the composite function, we can turn off the Y1 and Y2 entries so that only the graph of Y3 is shown. We can move the cursor to the “=” sign beside Y1 and Y2, and press the key to deactivate these graphs. On the screen, notice that only the composite function is now active.

Remember to select an appropriate viewing window for the graph. Evaluating the function when $x = -4$, we get:

Thus, $(f \circ g)(-4) = 131.$
We can also find \((f \circ g)(-4)\) on the home screen. Once we have entered the given functions into \(Y_1, Y_2\), we go to the home screen and enter \(Y_1(Y_2(-4))\) followed by \(\text{ENTER}\).

Calculating \((f \circ g)(-4)\) algebraically, we have \(g(-4) = 4(-4) + 3 = -16 + 3 = -13\) and \(f(-13) = (-13)^2 + 3(-13) + 1 = 169 - 39 + 1 = 131\).

### 5.2 IN-CLASS PRACTICE

1. Let \(f(x) = 2x^2 + x\) and \(g(x) = 3x - 1\). Find each function and state its domain.
   a. \((f + g)(x)\)
   b. \((f - g)(x)\)
   c. \((f \cdot g)(x)\)
   d. \(\left(\frac{f}{g}\right)(x)\)

2. Given \(f(x) = x^2 + 4\) and \(g(x) = \sqrt{x - 5}\). Find each of the following.
   a. \((f - g)(9)\)
   b. \((f \cdot g)(-3.5)\)
   c. \(\left(\frac{f}{g}\right)(21)\)

3. Members of a college drama club are working on a fundraising project to raise funds for new elaborate costumes for a new production. The fundraising includes hot dog, pie, and soda combo sales. The cost of the project can be approximated by \(C(x) = 0.08x^2 - 24x + 48\) and the expected revenue generated for \(x\) food items sold is \(R(x) = 4.50x\).
   a. Find \(R(x) - C(x)\) and explain what it means.
   b. Find and interpret \(P(200)\).

4. Use the graph to evaluate each function. State any undefined expression.
   a. \((f - g)(0)\)
   b. \(\left(\frac{f}{g}\right)(-3)\)
   c. \(\left(\frac{g}{f}\right)(2)\)
   d. \((f \circ g)(0)\)

5. Given \(f(x) = 4x + 5\), \(g(x) = \frac{x + 4}{x - 4}\), \(h(x) = \sqrt{x}\), and \(r(x) = x^2\), find the following and state the domain of each composite function.
   a. \((f \circ g)(x)\)
   b. \((g \circ f)(x)\)
   c. \((f \circ f)(x)\)
   d. \((r \circ h)(x)\)
6. Given \( f(x) = 3x + 4 \) and \( g(x) = \frac{x - 6}{4} \) find \((f \circ g)(3)\).

7. Pierre wants to buy a folding kayak for his summer vacations. The kayak he wants has a regular price of $\text{dollars}$ but it is being sold with a rebate of $350 and a 25% discount.
   a. Find a function, \( K(x) \), for the price of the kayak after the rebate.
   b. Find a function, \( D(x) \), for the price of the kayak after the discount.
   c. Find \((K \circ D)(x)\).
   d. Find and interpret \((K \circ D)(1595)\).
   e. Find \((D \circ K)(x)\).
   f. Find and interpret \((D \circ K)(1595)\).
   g. Which composite function is the better option for Pierre, \( K \circ D \) or \( D \circ K \)? Explain your decision.

8. Use the tables of \( f(x) \) and \( g(x) \) to complete the tables of values for \((f \circ g)(x)\)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>((f \circ g)(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>3</td>
<td>-3</td>
<td>-9</td>
</tr>
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<tr>
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<td>11</td>
<td>121</td>
</tr>
</tbody>
</table>

5.2 Exercises

**In exercises 1–8, find the following.**

a. \((f + g)(x)\)

b. \((f - g)(x)\)

c. \((f \cdot g)(x)\)

d. \(\left(\frac{f}{g}\right)(x)\)

1. \( f(x) = 3x - 9 \) and \( g(x) = -11x + 13 \)
2. \( f(x) = -7x + 12 \) and \( g(x) = 2x - 5 \)
3. \( f(x) = -3x - 8 \) and \( g(x) = 4x + 1 \)
4. \( f(x) = x^2 + 2x \) and \( g(x) = x - 1 \)
5. \( f(x) = 2x^2 - 3x \) and \( g(x) = 2x + 3 \)
6. \( f(x) = 3x - 7 \) and \( g(x) = 4x^2 - 2 \)
7. \( f(x) = 6x^2 + 2x \) and \( g(x) = 9x^2 - 1 \)
8. \( f(x) = 4x^2 - 9 \) and \( g(x) = 10x + 15 \)

**In exercises 9–12, find the following and state the domain.**

a. \((f + g)(x)\)

b. \((f - g)(x)\)

c. \((f \cdot g)(x)\)

d. \(\left(\frac{f}{g}\right)(x)\)

9. \( f(x) = 2x^2 - 5 \) and \( g(x) = x + 4 \)
10. \( f(x) = 6x^2 + 1 \) and \( g(x) = 2x + 9 \)
11. \( f(x) = 4x^3 - 5 \) and \( g(x) = 2x + 7 \)
12. \( f(x) = -3x - 4 \) and \( g(x) = x^2 - 16 \)

**In exercises 13–20, find each of the following and state the domain for each composite function.**

a. \((f \circ g)(x)\)

b. \((g \circ f)(x)\)

c. \((h \circ f)(x)\)

d. \((f \circ h)(x)\)

13. \( f(x) = 9x + 2 \), \( g(x) = x^2 - 2 \), and \( h(x) = \frac{4}{x + 3} \)
14. \( f(x) = -7x - 3 \), \( g(x) = x^2 + 5 \), and \( h(x) = \frac{5}{x - 8} \)
15. \( f(x) = 2x^2 - 1 \), \( g(x) = 5x - 2 \), and \( h(x) = \frac{8x - 5}{3} \)
16. \( f(x) = -4x - 5 \), \( g(x) = 3x^2 + 8x \), and \( h(x) = \frac{-2}{3x + 1} \)
17. \( f(x) = -6x + 2 \), \( g(x) = 2x^2 - 4x \), and \( h(x) = \frac{-7}{9x - 2} \)
18. \( f(x) = 3x + 4 \), \( g(x) = -5x^2 - 3 \), and \( h(x) = \sqrt{4x - 5} \)
19. \( f(x) = 8x - 9 \), \( g(x) = -6x^2 + 10 \), and \( h(x) = \sqrt{2x + 11} \)
20. \( f(x) = -5x - 2 \), \( g(x) = 4x^2 + 7 \), and \( h(x) = \sqrt{8 - 3x} \)

In exercises 21–26, find each of the following.

a. \((f \circ g)(x)\)  
b. \((g \circ f)(x)\)

21. \( f(x) = x^2 + 2 \) and \( g(x) = \sqrt{7 - 2x} \)
22. \( f(x) = 3x^2 - 5 \) and \( g(x) = \sqrt{4x + 7} \)
23. \( f(x) = x^2 - 8 \) and \( g(x) = 2\sqrt{3x + 1} \)
24. \( f(x) = 4x^2 - 3x + 5 \) and \( g(x) = \sqrt{2x - 1} \)
25. \( f(x) = 5x^2 + 2x - 1 \) and \( g(x) = 2\sqrt{x + 7} \)
26. \( f(x) = 7x^2 - 4x - 6 \) and \( g(x) = 3\sqrt{4x - 2} \)

In exercises 27–33, find the following for the given functions.

27. \( f(x) = 2x^2 - 7x \) and \( g(x) = 3x + 8 \)
   a. \((f + g)(-3)\)
   b. \((f - g)(-2)\)
   c. \((f \cdot g)(-4)\)
   d. \(\left(\frac{f}{g}\right)(-1)\)
28. \( f(x) = 3x^2 - 4x \) and \( g(x) = -4x + 5 \)
   a. \((f + g)(2)\)
   b. \((f - g)(-3)\)
   c. \((f \cdot g)(3)\)
   d. \(\left(\frac{f}{g}\right)(-1)\)
29. \( f(x) = -2x^2 + x - 7 \) and \( g(x) = -8x + 10 \)
   a. \((f + g)(-6)\)
   b. \((f - g)(-4)\)
   c. \((f \cdot g)(1)\)
   d. \(\left(\frac{f}{g}\right)(7)\)
30. \( f(x) = -4x^2 - 6x + 10 \) and \( g(x) = 8x + 30 \)
    a. \((f + g)(4)\)
    b. \((f - g)(-5)\)
    c. \((f \cdot g)(-4)\)
    d. \(\left(\frac{f}{g}\right)(-9)\)
31. \( f(x) = \sqrt{2x + 2} \) and \( g(x) = -2x + 11 \)
    a. \((f + g)(7)\)
    b. \((f - g)(17)\)
    c. \((f \cdot g)(1)\)
    d. \(\left(\frac{f}{g}\right)(31)\)
32. \( f(x) = \sqrt{4x + 1} \) and \( g(x) = x - 12 \)
    a. \((f + g)(12)\)
    b. \((f - g)(6)\)
    c. \((f \cdot g)(20)\)
    d. \(\left(\frac{f}{g}\right)(12)\)
33. \( f(x) = \sqrt{6x - 5} \) and \( g(x) = 3x - 9 \)
    a. \((f + g)(5)\)
    b. \((f - g)(1)\)
    c. \((f \cdot g)(6)\)

In exercises 34–41, use your calculator to find the following, and then leave your answers in decimal form. Round to 2 decimals places where needed.

34. \( f(x) = \frac{3}{2x + 8} \) and \( g(x) = 4x + 5 \)
    a. \((f + g)(6)\)
    b. \((f - g)(-4)\)
    c. \((f \cdot g)(7)\)
    d. \(\left(\frac{f}{g}\right)(0.75)\)
35. \( f(x) = \frac{-6}{7x - 9} \) and \( g(x) = -2x + 3 \)
    a. \((f + g)(\frac{7}{5})\)
    b. \((f - g)(1.46)\)
    c. \((f \cdot g)(\frac{9}{7})\)
    d. \(\left(\frac{f}{g}\right)(0)\)
36. \( f(x) = \frac{-1.9}{5.8x - 1} \) and \( g(x) = -6.1x - 3.2 \)
    a. \((f + g)(3.2)\)
    b. \((f - g)(\frac{7}{8})\)
    c. \((f \cdot g)(-3)\)
    d. \(\left(\frac{f}{g}\right)(-0.65)\)
37. \( f(x) = \frac{-21x}{0.2x + 2} \) and \( g(x) = 5x + 0.8 \)
    a. \((f \circ g)(2)\)
    b. \((g \circ f)(7)\)
38. \( f(x) = \frac{31x}{5.3x + 4.2} \) and \( g(x) = -6x - 1.8 \)
    a. \((f \circ g)(0)\)
    b. \((g \circ f)(-9.9)\)
39. \( f(x) = 3.3x^2 - 2x + 4 \) and \( g(x) = \sqrt{4.5x - 7} \)
    a. \((f \circ g)(6.1)\)
    b. \((g \circ f)(-7.3)\)
40. \( f(x) = -0.83x^2 + 12x - 1.4 \) and 
\( g(x) = \sqrt{2.9 - 0.3x} \)

a. \( (f \circ g)(1.9) \)
b. \( (g \circ f)(0) \)
c. \( (f \cdot g)(-5) \)
d. \( \left( \frac{1}{g} \right)(-5) \)

41. \( f(x) = \frac{4x - 1}{3.5x} \) and 
\( g(x) = \sqrt{7x - 1.65} \)

a. \( (f \circ g)(10) \)
b. \( (g \circ f)(-18) \)

d. \( f(x) = -0.83x^2 + 12x - 1.4 \) and 
\( g(x) = \sqrt{2.9 - 0.3x} \)

\( (f \circ g)(1.9) \), \( (g \circ f)(0) \), \( (f \cdot g)(-5) \), \( \left( \frac{1}{g} \right)(-5) \)

42. a. \( (f + g)(0) \)
b. \( (f - g)(-5) \)
c. \( (f \cdot g)(0) \)
d. \( \left( \frac{1}{g} \right)(-5) \)

e. \( (f \circ g)(9) \)

43. a. \( (f + g)(4) \)
b. \( (f - g)(1) \)
c. \( (f \cdot g)(0) \)
d. \( (g \circ f)(-2) \)
e. \( (f \circ g)(3) \)

44. a. \( (f + g)(-6) \)
b. \( (f - g)(-3) \)
c. \( (f \cdot g)(-2) \)
d. \( (g \circ f)(-9) \)
e. \( (f \circ g)(-3) \)

45. a. \( (f + g)(-3) \)
b. \( (f - g)(-5) \)
c. \( (f \cdot g)(-7) \)
d. \( (g \circ f)(-4) \)
e. \( (f \circ g)(-1) \)

46. a. \( (f + g)(-4) \)
b. \( (f - g)(0) \)
c. \( (f \cdot g)(0) \)
d. \( (g \circ f)(-5) \)
e. \( (f \circ g)(2) \)

47. a. \( (f + g)(-4) \)
b. \( (f - g)(0) \)
c. \( (f \cdot g)(4) \)
d. \( (g \circ f)(4) \)
e. \( (f \circ g)(2) \)

48. A company manufactures and sells DVD stands for homes and offices. The revenue in dollars for the company for \( x \) DVD stands sold is modeled by \( R(x) = x(65 - 0.01x) \). The total cost of producing the stands involves a fixed cost of $600, plus a variable cost of $15 per unit to be sold.

a. Write the total cost function \( C(x) \), for this problem.

b. What is the profit function, \( P(x) \)?

c. Find and interpret \( P(250) \).

49. Mr. Washington is an eleventh grade high school math teacher who constantly has to remind his students that cell phones are not to be used in class. He decides to give them an assignment in which they create a word problem that involves cell phones and asks them to include some research on the history of the cell phone itself. Johnny creates a word problem in which a company’s revenue in dollars for \( x \) number of refurbished cell phones sold is modeled by \( R(x) = x(80 - 0.01x) \). The total cost of refurbishing the cell phones involves a fixed cost of $450 plus a variable cost of $20 per cell phone.

**Little Facts:** In 1973, Dr. Martin Cooper, an electric engineer and employee at Motorola, changed the way people communicate via phone. AT&T had already created cellular technology but it was limited to the car. Mr. Cooper and his team launched the first portable mobile handset ten years later, called the DynaTAC. He and his wife are the inventors of the “Jitterbug” cell phone, geared towards senior citizens, which requires no contract or prepaid fees. In 1928, Paul Galvin and his brother Joseph...
50. Mrs. Washington is a twelfth grade accounting teacher who has given her students an assignment that involves creating a fictitious company which includes research on how to form a corporation. Part of the assignment is to create revenue and cost functions in which they will come up with ideas on how to reduce costs. Their company’s revenue in dollars for x number of units sold is modeled by \( R(x) = x(330 - 0.01x) \). Their total cost of producing x number of units involves a fixed cost of $2800 plus a variable cost of $32 per unit.

a. Write the total cost function \( C(x) \), for this problem.

b. What is the profit function, \( P(x) \)?

c. Find and interpret \( P(50) \).

51. A tennis team travels to Dubai and is having a contest to see whose group can launch the highest tennis ball from the 1,483 foot observatory deck of the Burj Khalifa, the world’s tallest building as of January 2011. The height in feet of team A’s tennis ball can be modeled by \( f(t) = -16t^2 + 73t + 1483 \) and the height of team B’s tennis ball can be modeled by \( g(t) = -16t^2 + 60t + 1483 \), where \( t \) represents time in seconds.

Little Facts: Dubai is one of the seven United Arab Emirates located in the in the Persian Gulf. As of Jan. 2011, the Burj Khalifa was the world’s tallest building, over 160 stories high and reaching 2,716.5 feet. Off the coast of Dubai is “The World,” a cluster of 300 manmade islands along with the “Palm Trilogy,” an island in the shape of a palm tree; both were created from sand dredged from the sea. Sources: www.burjkhalifa.ae; www.hotliermiddleeast.com; www.dubaitourism.ae

a. Find \( f(5.5) \) and interpret its meaning.

b. Find \( f(g)(5.5) \) and interpret its meaning.

c. Find \( f - g(5.5) \) and interpret its meaning.

d. Find \( f - g(2) \) and interpret its meaning.

52. Liana is a theatre major who does Beyoncé impersonations, then posts them on YouTube. The number of hits she received on YouTube over an 11 month period can be approximated by \( f(n) = -32.17n^2 + 395.17n - 161.27 \) where \( n \) denotes the number of months. Her sister Nicole does Avril Lavigne impersonations and the number of hits she receives can be approximated by \( g(n) = -29.2n^2 + 330.5n - 100 \) where \( n \) denotes the number of months.

a. Find \( f + g(n) \) and interpret its meaning.

b. Find \( f + g(6) \) to the nearest whole number and interpret its meaning.

c. Find \( f - g(n) \) and interpret its meaning.

d. Find \( f - g(10) \) to the nearest whole number and interpret its meaning.

53. A computer store is having a year-end special on all laptop computers by offering a rebate of $50 plus a 10% discount. Find the following if the suggested retail price of the computers is \( p \) dollars.

a. Write a function, \( R(p) \), that represents the cost of a laptop computer after the store’s rebate.

b. Write a function, \( D(p) \), that represents the cost of a laptop computer after the 10% discount.

c. Find and interpret \( (R \circ D)(p) \).

d. Find and interpret \( (R \circ D)(650) \).

e. Find and interpret \( (D \circ R)(p) \).

f. Find and interpret \( (D \circ R)(650) \).

g. Which composite function is the better option for the consumer, \( R \circ D \) or \( D \circ R \)? Explain your decision.

54. A coat store is having a close-out sale on all the winter clothing. They are offering an in-store $10.00 coupon plus a 35% discount on all winter coats. Find the following if the suggested retail price of the winter jackets is \( p \) dollars.

a. Write a function, \( R(p) \), that represents the cost of a winter jacket after the in-store coupon.

b. Write a function, \( D(p) \), that represents the cost of a winter jacket after the 35% discount.

c. Find and interpret \( (R \circ D)(p) \).

d. Find and interpret \( (R \circ D)(150) \).

e. Find and interpret \( (D \circ R)(p) \).

f. Find and interpret \( (D \circ R)(150) \).

g. Which composite function is the better option for the consumer, \( R \circ D \) or \( D \circ R \)? Explain your decision.
Reflective Thought

55. True or False. If \( f(x) = g(x) \) then \( (f \circ g)(x) = (g \circ f)(x) \). Explain your answer.

56. Kerry was given the following functions: \( f(x) = x + 2 \) and \( g(x) = x - 9 \), and was asked to find \( (f \circ g)(2) \). She found that \( f(2) = 4 \) and \( g(4) = -5 \) and concluded that \( (f \circ g)(2) = -5 \). Is she correct or incorrect, explain why or why not.

57. The following tables display values for two functions \( f(x) \) and \( g(x) \) respectively. Use the data from the tables to find \( (f \circ g)(3) \) and \( (g \circ f)(2) \).

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Table 2</th>
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<tbody>
<tr>
<td>( x )</td>
<td>( f(x) )</td>
</tr>
<tr>
<td>2</td>
<td>-11</td>
</tr>
<tr>
<td>8</td>
<td>-41</td>
</tr>
<tr>
<td>9</td>
<td>-46</td>
</tr>
<tr>
<td>13</td>
<td>-66</td>
</tr>
<tr>
<td>15</td>
<td>-76</td>
</tr>
</tbody>
</table>

**Answer Key**

5.2 Exercises

1. a. \(-8x + 4\)
   b. \(14x - 22\)
   c. \(-33x^2 + 138x - 117\)
   d. \(\frac{3x - 9}{-11x + 13}\)
2. a. \(15x^2 + 2x - 1\)
   b. \(-3x^2 + 2x + 1\)
   c. \(54x^4 + 18x^3 - 6x^2 - 2x\)
   d. \(\frac{2x}{3x - 1}\)

3. a. \(x - 7\)
   b. \(-7x - 9\)
   c. \(-12x^2 - 35x - 8\)
   d. \(\frac{-3x - 8}{4x + 1}\)
4. a. \(2x^3 - 5x - 3\)
   b. \(2x^3 - 3x\)
   c. \(2x^3 - 9x\)
   d. \(\frac{2x^3 - 3x}{2x + 3}\)

5. a. \(2x^2 - x + 3\)
   b. \(2x^2 - 5x - 3\)
   c. \(4x^3 - 9x\)
   d. \(2x^2 - 3x\)

6. a. \(4x^3 - 3x + 2; (-\infty, \infty)\)
   b. \(4x^3 - 2x - 12; (-\infty, \infty)\)
   c. \(8x^4 + 28x^3 - 10x - 35; (-\infty, \infty)\)
   d. \(4x^3 - 5; \left(-\infty, -\frac{7}{2}\right)\cup\left(-\frac{7}{2}, \infty\right)\)

7. a. \(9x^2 - 16; (-\infty, \infty)\)
   b. \(81x^2 + 36x + 2; (-\infty, \infty)\)
   c. \(\frac{4}{9x + 5}\)
   d. \(\frac{36}{x + 3} + 2\) or \(\frac{2x + 42}{x + 3}\;\;(-\infty, -3)\cup(-3, \infty)\) or \(x \neq -3\)

8. a. \(50x^2 - 40x + 7; (-\infty, \infty)\)
   b. \(10x^2 - 7; (-\infty, \infty)\)
   c. \(\frac{16x^2 - 13}{3}; (-\infty, \infty)\)

9. a. \(-12x^2 + 24x + 2; (-\infty, \infty)\)
   b. \(72x^2 - 24x; (-\infty, \infty)\)
   c. \(\frac{-7}{-54x + 16}\)
   d. \(\frac{42}{9x - 2} + 2\) or \(\frac{18x + 38}{9x - 2}\)

10. a. \(-48x^2 + 71; (-\infty, \infty)\)
    b. \(-384x^2 + 864x - 476; (-\infty, \infty)\)
    c. \(\sqrt{16x - 7}; \left[\frac{7}{16}\right]^{\infty}\) or \(x \geq \frac{7}{16}\)
    d. \(8\sqrt{2x + 11} - 9; \left[-\frac{11}{2}, \infty\right)\) or \(x \geq -\frac{11}{2}\)

11. a. \(-2x + 9\)
   b. \(\sqrt{-2x^2 + 3}\)
   c. \(\sqrt{16x - 7}; \left[\frac{7}{16}\right]\)
   d. \(8\sqrt{2x + 11} - 9; \left[-\frac{11}{2}, \infty\right)\)

12. a. \(12x - 4\)
   b. \(2\sqrt{3x^2 - 23}\)
   c. \(20x + 4\sqrt{x + 7} + 139\)
   d. \(2\sqrt{5x^2 + 2x + 6}\)
5.3 Determining the Inverse Using Composition of Functions

Section 5.3

Determining the Inverse Using Composition of Functions

Recall from Section 5.1 that the inverse of a one-to-one function is simply the set of ordered pairs obtained when we swap the $x$ and $y$ coordinates in the original function, that is, when we reverse the action of the function. Let us review by finding the inverse function of $f(x) = 7x$.

$$f(x) = 7x \rightarrow y = 7x \rightarrow x = \frac{y}{7}$$

So, $f^{-1}(x) = \frac{x}{7}$.

Notice how $f$ and $f^{-1}$ “undo” each other. The rule in $f$ is to “multiply 7 times the input,” whereas $f^{-1}$ says “divide the input by 7.”

Suppose $x = 3$ is an input for $f(x)$. Then $f(3) = 7(3) = 21$. So, $f$ yields the ordered pair $(3, 21)$. Evaluating $f^{-1}(21) = \frac{21}{7} = 3$, now $f^{-1}$ generates the ordered pair $(21, 3)$. We can see how in the function $f$ we started with 3 and got 21, whereas in the inverse function $f^{-1}$ we started with 21 and we arrived back at 3. Summarizing, $f(3) = 21$ and $f^{-1}(21) = 3$.
In general, if \( f^{-1} \) is the inverse of \( f \), then if \( f(a) = b \), we know that \( f^{-1}(b) = a \). That is, the inverse gets us back to where we started.

Since \( f(a) = b \), notice that \( f^{-1}(b) \) is equivalent to finding \( f^{-1}(f(a)) \), which is the composite function of \( f^{-1} \) and \( f \). On the other hand, since \( f^{-1}(b) = a \), notice that \( f(a) \) is equivalent to finding \( f(f^{-1}(b)) \), which is the composite function of \( f \) and \( f^{-1} \).

The basic idea is that processes an output from \( f \) and returns it back to the original input in \( f \), while \( f^{-1} \) processes an output from \( f^{-1} \) and returns it back to the original input in \( f^{-1} \). The following alternate definition of inverse functions is useful to determine if two functions are inverses of each other.

### Inverse Functions

Two functions \( f \) and \( g \) are inverses of each other if and only if

\[
(f \circ g)(x) = f(g(x)) = x \text{ for every } x \text{ in the domain of } g
\]

and

\[
(g \circ f)(x) = g(f(x)) = x \text{ for every } x \text{ in the domain of } f.
\]

So, \( g = f^{-1} \) and \( f = g^{-1} \).

The domain of \( f \) is equivalent to the range of \( g \) and vice versa.

### EXAMPLE 1

Show that \( f(x) = 7x \) and \( g(x) = \frac{x}{7} \) are inverses of each other.

**Solution**

First, if \( g(x) \) is the inverse of \( f(x) \), then \( (f \circ g)(x) = x \).

\[
(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{7}\right) = 7\left(\frac{x}{7}\right) = x. \quad \text{It checks.}
\]

Now, if \( f(x) \) is the inverse of \( g(x) \), then \( (g \circ f)(x) = x \).

\[
(g \circ f)(x) = g(f(x)) = g(7x) = \frac{7x}{7} = x. \quad \text{It checks.}
\]

Therefore, \( f \) and \( g \) are inverses of each other.

### EXAMPLE 2

Determine whether the given functions \( f \) and \( g \) are inverses of each other.

a. \( f(x) = \frac{6}{x - 3} \) and \( g(x) = \frac{6}{x} + 3 \)  

b. \( f(x) = \frac{1}{2x + 5} \) and \( g(x) = 2x + 5 \)

**Solution**

a. We will find \( f \circ g \) and \( g \circ f \) and see if the result is \( x \) for both compositions.

\[
(f \circ g)(x) = f(g(x)) = f\left(\frac{6}{x} + 3\right) = \frac{6}{\left(\frac{6}{x} + 3\right) - 3} = \frac{6}{\frac{6}{x}} = 6 \cdot \frac{x}{6} = x.
\]

\[
(g \circ f)(x) = g(f(x)) = g\left(\frac{6}{x - 3}\right) = \frac{6}{\left(\frac{6}{x - 3}\right)} + 3 = 6 \cdot \frac{x - 3}{6} + 3 = x - 3 + 3 = x.
\]

So, \( g = f^{-1} \) and \( f = g^{-1} \).
5.3 Determining the Inverse Using Composition of Functions

b. Let us begin by finding \( f \circ g \).

\[
(f \circ g)(x) = f(g(x)) = f(2x + 5) = \frac{1}{2(2x + 5) + 5} = \frac{1}{4x + 15} \neq x.
\]

Since \( f \circ g \neq x \), there is no need to find \( g \circ f \). Functions \( f \) and \( g \) are not inverses of each other. As we examine the given functions \( f(x) = \frac{1}{2x + 5} \) and \( g(x) = 2x + 5 \), it is convenient to recall that \( f^{-1}(x) \neq \frac{1}{f(x)} \).

**EXAMPLE 3**

Find the inverse function of \( f(x) = \sqrt{x + 6} \) for \( x \geq -6 \), and state the domain for both \( f \) and \( f^{-1} \). Verify algebraically and graphically that \( f \) and \( f^{-1} \) are inverses of each other.

**Solution**

The inverse of \( f(x) = \sqrt{x + 6} \) for \( x \geq -6 \) is calculated next.

\[
f(x) = \sqrt{x + 6} \\
y = \sqrt{x + 6} \\
x = \sqrt{y + 6} \\
x^2 = (\sqrt{y + 6})^2 \\
x^2 = y + 6 \\
\implies y = x^2 - 6
\]

So, \( f^{-1}(x) = x^2 - 6 \).

The domain of \( f \) is \([-6, \infty)\) and the range is \([0, \infty)\). We know that the domain of \( f^{-1} \) is the same as the range of \( f \), and the range of \( f^{-1} \) is the same as the domain of \( f \). Thus, the domain of \( f^{-1} \) is \([0, \infty)\) and its range is \([-6, \infty)\).

Now we find \( f \circ f^{-1} \) and \( f^{-1} \circ f \) to verify that the result is \( x \) for both compositions.

\[
(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(x^2 - 6) = \sqrt{(x^2 - 6) + 6} = \sqrt{x^2} = x.
\]

\[
(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(\sqrt{x + 6}) = \left(\sqrt{x + 6}\right)^2 - 6 = x + 6 - 6 = x.
\]

We previously learned that the graphs of a function and its inverse are symmetric about the line \( y = x \). The graphs of \( f \) and \( f^{-1} \), along with \( y = x \) are shown.

Notice that both functions are symmetric with respect to \( y = x \).
EXAMPLE 4
As of 2010, the sales tax rate for taxable goods or services in Seminole County, Florida was 7%. Source: www.seminolecountyfl.gov

a. Find the function \( T(p) \) that represents the sales tax for price \( p \).
b. Determine \( T^{-1}(p) \) and explain its meaning in the context of this problem.
c. Evaluate and interpret \( T^{-1}(5.25) \).
d. Show that \( T \) and \( T^{-1} \) are inverses of each other.

Solution
a. The sales tax is calculated by multiplying the price of the item by 7%. So, \( T(p) = 0.07p \).
b. The inverse of \( T(p) = 0.07p \) is given next.

\[
T(p) = 0.07 \rightarrow y = 0.07p \rightarrow p = 0.07y \rightarrow y = \frac{p}{0.07} \quad \text{So, } T^{-1}(p) = \frac{p}{0.07}.
\]

The inverse gives the price of an item or service if \( p \) dollars are paid as sales tax.

c. \( T^{-1}(5.25) = \frac{5.25}{0.07} = 75 \). If the sales tax is $5.25, the price is $75.

d. \( (T \circ T^{-1})(p) = T(T^{-1}(p)) = T\left(\frac{p}{0.07}\right) = 0.07\left(\frac{p}{0.07}\right) = p \).

\[
(T^{-1} \circ T)(p) = T^{-1}(T(p)) = T^{-1}(0.07p) = \left(\frac{0.07p}{0.07}\right) = p.
\]

Since the result is \( p \) for both compositions, \( T \) and \( T^{-1} \) are inverses of each other.

EXAMPLE 5
The graph of a function, \( f(x) \), is shown.
a. Graph \( f^{-1}(x) \) along with the line \( y = x \).
b. Find \( (f \circ f^{-1})(2) \).
c. Find \( (f^{-1} \circ f)(2) \).
5.3 Determining the Inverse Using Composition of Functions

Solution

a. We know that the graphs of a function and its inverse are symmetric about the line $y = x$, thus we can use this symmetry to graph the inverse function. The points $(0, 1), (1, 2), (2, 4),$ and $(3, 8)$ lie on the graph of $f(x)$. Therefore, the points $(1, 0), (2, 1), (4, 2),$ and $(8, 3)$ will lie on the graph of $f^{-1}(x)$.

Plotting and connecting these points with a smooth curve provides us with the graph.

b. $(f \circ f^{-1})(2) = f(f^{-1}(2)) = f(1) = 2$.  
c. $(f^{-1} \circ f)(2) = f^{-1}(f(2)) = f^{-1}(4) = 2$.

On each case, we can see that when $x = 2$, the composition brought us back to the value of $x$.

EXAMPLE 6

Vijay went bungee jumping with his friends on his 30th birthday. The bridge from which he jumped was 250 feet above the surface of the river. The height in feet, $h$, of Vijay as he jumped and then dived into the water is given by the function $h(x) = -16x^2 + 250$, where $x$ is time in seconds.

a. Is $h(x)$ a one-to-one function? Explain your decision.

b. Find and interpret $h^{-1}(x)$.

c. Find and interpret $h^{-1}(150)$.

Solution

a. Since $x$ represents time in seconds, $x \geq 0$ and the quadratic function will have a restricted domain; it will pass the horizontal line test. Hence, $h(x)$ is a one-to-one function.
In a hot air balloon race, the height of Team Grand Nationals’ balloon at minutes is given by 
\[ h(t) = -16t^2 + 250 \]

In exercises 1–6, determine whether the given functions \( f \) and \( g \) are inverses of each other.

1. Determine whether the given functions \( f \) and \( g \) are inverses of each other.
   a. \( f(x) = 3x + 9; \ g(x) = \frac{x - 9}{3} \)
   b. \( f(x) = \frac{1}{x + 14}; \ g(x) = x + 14 \)
   c. \( f(x) = \frac{1}{x}; \ g(x) = \frac{1}{x} \)

2. a. Find the inverse function of \( f(x) = \sqrt{2x + 1} \) for \( x \geq -\frac{1}{2} \).
   b. State the domain for both \( f \) and \( f^{-1} \).
   c. Verify algebraically and graphically that \( f \) and \( f^{-1} \) are inverses of each other.

3. In a hot air balloon race, the height \( h \) of Team Grand Nationals’ balloon at \( t \) minutes is given by \( h(t) = 30 + 50t \).
   a. Find \( h^{-1}(t) \).
   b. Find and interpret \( h^{-1}(355) \).
   c. Verify algebraically that \( h \) and \( h^{-1} \) are inverses of each other.

### 5.3 IN-CLASS PRACTICE

In exercises 1–6, determine whether the given functions \( f \) and \( g \) are inverses of each other.

1. \( f(x) = -2x + 3 \) and \( g(x) = \frac{3 - x}{2} \)
2. \( f(x) = -8x + 5 \) and \( g(x) = \frac{5}{8} - x \)
3. \( f(x) = \frac{1}{4x - 7} \) and \( g(x) = \frac{x + 7}{4} \)
4. \( f(x) = \frac{-1}{3x + 5} \) and \( g(x) = \frac{-1}{3x} - \frac{5}{3} \)
5. \( f(x) = \frac{2x - 1}{3} \) and \( g(x) = \frac{3x + 1}{2} \)
6. \( f(x) = \frac{-4x + 1}{7} \) and \( g(x) = \frac{7x + 1}{4} \)

In exercises 7–12, find the following.

1. The inverse of the function \( f(x) \).
2. The domain and the range for both \( f(x) \) and \( f^{-1}(x) \).
3. Graph both \( f(x) \) and \( f^{-1}(x) \); label axes and tick marks.

7. \( f(x) = \sqrt{x + 3} \) for \( x \geq -3 \)
8. \( f(x) = \sqrt{x - 4} \) for \( x \geq 4 \)
9. \( f(x) = \sqrt{2x - 5} \) for \( x \geq \frac{5}{2} \)
10. \( f(x) = \sqrt{3x + 1} \) for \( x \geq -\frac{1}{3} \)
11. \( f(x) = \sqrt{\frac{x + 1}{2}} \) for \( x \geq -1 \)
12. \( f(x) = \sqrt{\frac{x - 8}{3}} \) for \( x \geq 8 \)
13. The graph of a function, $f(x)$, is shown.
   a. Graph $f^{-1}(x)$ along with the line $y = x$. Use the given plotted points as a guide.
   b. Find $(f^{-1} \circ f)(4)$ and plot the points.
   c. Find $(f \circ f^{-1})(4)$ and plot the points.

14. The graph of a function, $f(x)$, is shown.
   a. Graph $f^{-1}(x)$ along with the line $y = x$. Use the given plotted points as a guide.
   b. Find $(f^{-1} \circ f)(4)$ and plot the points.
   c. Find $(f \circ f^{-1})(4)$ and plot the points.

15. The graph of a function, $f(x)$, is shown.
   a. Graph $f^{-1}(x)$ along with the line $y = x$. Use the given plotted points as a guide.
   b. Find $(f^{-1} \circ f)(5)$ and plot the points.
   c. Find $(f \circ f^{-1})(5)$ and plot the points.
16. The graph of a function, \( f(x) \), is shown.
   a. Graph \( f^{-1}(x) \) along with the line \( y = x \). Use the given plotted points as a guide.
   b. Find \( (f^{-1} \circ f)(6) \) and plot the points.
   c. Find \( (f \circ f^{-1})(6) \) and plot the points.

17. The graph of a function, \( f(x) \), is shown.
   a. Graph \( f^{-1}(x) \) along with the line \( y = x \). Use the given plotted points as a guide.
   b. Find \( (f^{-1} \circ f)(3) \) and plot the points.
   c. Find \( (f \circ f^{-1})(3) \) and plot the points.

18. The graph of a function, \( f(x) \), is shown.
   a. Graph \( f^{-1}(x) \) along with the line \( y = x \). Use the given plotted points as a guide.
   b. Find \( (f^{-1} \circ f)(6) \) and plot the points.
   c. Find \( (f \circ f^{-1})(6) \) and plot the points.

19. A custom golf shirts company charges a design fee of $15 and sells each shirt for $35. The total cost, \( f(x) \), can be represented by the function \( f(x) = 35x + 15 \), where \( x \) represents the number of golf shirts.
   a. Find and interpret \( f^{-1}(x) \).
   b. Find and interpret \( f^{-1}(365) \).
20. Rachel and her friend agree to go jogging around Lake Eola in Orlando, Florida. After jogging around the lake, they decide to rent a swan boat and talk about their day at work. They paid a deposit of $15 to rent a boat, which costs $24.00 per hour. The total cost, \( C(x) \), to rent the boat can be modeled by the function \( C(x) = 24x + 15 \), where \( x \) represents the number of hours they used the boat.

**Little Facts:** Charles Lord, a wealthy businessman from England arrived in Orlando, Florida in 1885. In 1910, Mr. Lord brought a pair of black and white swans from England and placed them in Lake Lucerne, where he had a spacious house. One of the male swans, “Billy,” constantly picked on the other swans, so the swans he was picking on were moved to Lake Eola. History has it that Billy the swan lived to be 75 years old. Sources: www.fox59.com; www.orlandosentinel.com

a. Find and interpret \( C(1.25) \).

b. Find and interpret \( C^{-1}(47) \).

c. Find and interpret \( C^{-1}(75) \) to 1 decimal place.

d. Verify algebraically that \( C \) and \( C^{-1} \) are inverses of each other.

21. Gerard is studying to be a dietician and is doing a research to determine how long it takes to burn off a meal eaten at a fast food restaurant. Gerard goes to McDonalds and orders a Big Mac, a medium order of fries, and a 16-ounce Coke. After researching the calorie intake from their website, he realizes that he has consumed a total of 1070 calories: 540 from the Big Mac, 380 from the fries and 150 from the Coke. He then decides to play basketball, which allows a typical 170 pound person to burn 584 calories per hour. The total calories remaining, \( R(x) \), after \( x \) hours of playing basketball can be represented by the function \( R(x) = 1070 - 584x \). Sources: www.mcdonalds.com; www.mayoclinic.com

a. Find and interpret \( R(0.75) \).

b. Find and interpret \( R^{-1}(5) \).

c. Find and interpret \( R^{-1}(100) \). Round your answer to 2 decimal places.

d. Verify algebraically that \( R \) and \( R^{-1} \) are inverses of each other.

22. Mark is studying to be a nutritionist and is doing a research to determine how long it takes to burn off a meal eaten at a fast food restaurant. Mark goes to McDonalds and orders a Quarter Pounder with cheese, a small order of fries and a 16-ounce Sprite. After researching the calorie intake from their website, he realizes that he has consumed a total of 910 calories: 510 from the Quarter Pounder with cheese, 250 from the fries and 150 from the Sprite. He then decides to go to the gym to do some low-impact aerobics, which allows a typical 170 pound person to burn 365 calories per hour. The total calories remaining, \( R(x) \), after \( x \) hours of playing basketball can be represented by the function \( R(x) = 910 - 365x \). Sources: www.mcdonalds.com; www.mayoclinic.com

a. Find and interpret \( R(1.2) \).

b. Find and interpret \( R^{-1}(2) \).

c. Find and interpret \( R^{-1}(50) \). Round your answer to 2 decimal places.

d. How long will it take Mark to burn off all of his McDonald calories? Round your answer to 2 decimal places.

e. Find and interpret \( h^{-1}(2) \) to 2 decimal places.

23. Vince went bungee jumping with his friends on his 25th birthday. The bridge from which he jumped was 175 feet above the surface of the river. The height in feet, \( h \), of Vince as he jumped and then dived into the water is given by the function \( h(x) = -16x^2 + 175 \), where \( x \) is time in seconds and \( x \geq 0 \).

a. Is \( h(x) \) a one-to-one function? Explain your decision.

b. Find and interpret \( h(2.8) \) to 2 decimal places.

c. Find and interpret \( h^{-1}(x) \).

d. Find and interpret \( h^{-1}(100) \) to 2 decimal places.

e. When will Vince and his friends reach the surface of the water? Round your answer to 2 decimal places.

f. Graph the function \( h(x) \), label the axes and tick marks.

g. Verify algebraically that \( h \) and \( h^{-1} \) are inverses of each other.

24. Oscar is a tourist from Atlanta, Georgia who decides to go to the Amway Center to watch the Orlando Magic play the Atlanta Hawks. He and his friends decide to go to the Gentleman Jack Bar to get a view of the city of Orlando, which is situated 180 feet above ground. Once outside, they drop a ten-dollar bill attached to a plastic weight to the crowd below. The height in feet, \( h \), of the ten-dollar bill is given by the function \( h(x) = -16x^2 + 180 \), where \( x \) is time in seconds.

**Little Facts:** With a price tag of approximately 480 million dollars and a seating capacity of 18,500, the Amway Center opened in Orlando, Florida in October of 2010. As of 2011, the Amway Center boasted the tallest video scoreboard measuring in at 42 feet by 41 feet. The scoreboard contains nine million LEDs and weighs approximately 40 tons. Sources: www.amwaycenter.com; www.bleacherreport.com; www.nba.com

a. Is \( h(x) \) a one-to-one function? Explain your decision.

b. Find and interpret \( h(3) \).

25. Find and interpret \( h^{-1}(x) \).
25. A roofer was applying waterproof material in preparation for the hurricane season. He accidentally dropped the roofing nail gun from a height of 100 feet. Its height in feet, \( h \), after \( x \) seconds is modeled by the function \( h(x) = -16x^2 + 100 \) for \( x \geq 0 \). (Don’t worry, luckily no one was hurt!)
   a. Find and interpret \( h(1.4) \). Round your answer to 2 decimal places.
   b. Find and interpret \( h^{-1}(x) \).
   c. Find and interpret \( h^{-1}(60) \) to 2 decimal places.
   d. When does the nail gun reach the ground?
   e. When will the ten-dollar bill reach the ground? Round your answer to 2 decimal places.
   f. Graph the function \( h(x) \), label the axes and tick marks.

Reflective Thought

26. Given \( f(x) = \sqrt{x + 13} \), give the restriction on the domain in order for the function to have an inverse. Explain why the restriction is needed and find the inverse.

27. Carly was given \( f(x) = \frac{2}{x - 6} \) and found the inverse, which is \( f^{-1}(x) = \frac{2}{x} + 6 \). She was then asked to find \( (f^{-1} \circ f)(6) \).
   Carly concluded that since the two functions are inverses of each other, then \( (f^{-1} \circ f)(x) = x \), therefore \( (f^{-1} \circ f)(6) = 6 \).
   Her professor gave her partial credit. Explain why Carly was not given full credit.

Answer Key 5.3 Exercises

1. Yes
3. No
5. Yes
7. a. \( f^{-1}(x) = x^2 - 3 \)
   b. \( f(x): D = [-3, \infty); R = [0, \infty); \) \( f^{-1}(x): D = [0, \infty); R = [-3, \infty) \)
   c. 

9. a. \( f^{-1}(x) = \frac{x^2 + 5}{2} \)
   b. \( f(x): D = \left[ \frac{5}{2}, \infty \right); R = [0, \infty); \)
   c. 

11. a. \( f^{-1}(x) = 2x^2 - 1 \)
   b. \( f(x): D = [-1, \infty); R = [0, \infty); \)
   c. 

13. a. 
   b. 4
   c. 4

15. a. 
   b. 5
   c. 5

17. a. 
   b. 3
   c. 3
19. a. \( f^{-1}(x) = \frac{x - 15}{35} \); The inverse gives us the number of golf shirts sold when the total cost is \( x \) dollars.
   b. \( f^{-1}(365) = 10 \); If the total cost is $365, 10 shirts were sold. Answers may vary.
21. a. 632; After playing basketball for 0.75 hours, 632 calories still remain. Answers may vary.
   b. \( R^{-1}(x) = \frac{1070 - x}{584} \); The inverse gives us the number of hours exercised given the \( x \) number of calories remaining.
       Answers may vary.
   c. \( R^{-1}(100) = 1.66 \); if 100 calories remain then Gerard played basketball for 1.66. Answers may vary.
   d. \( R(R^{-1}(x)) = x \) and \( R^{-1}(R(x)) = x \)
23. a. Since \( x \) represents time in seconds, \( x \geq 0 \) and the quadratic function will have a restricted domain, and it will pass the horizontal line test. Hence, \( h(x) \) is a one-to-one function. Answers may vary.
   b. \( h(2.8) = 49.56 \); In 2.8 seconds, Vince and his friends are 49.56 feet above the water.
   c. \( h^{-1}(x) = \sqrt{\frac{175 - x}{16}} = \sqrt{\frac{175 - x}{4}} \); The inverse gives us the time is seconds when Vince and his friends are at a height of \( x \) feet. Answers may vary.
   d. 2.17 When Vince and his friends are 100 feet above the water, 2.17 seconds elapsed. Answers may vary.
   e. 3.31 seconds

25. a. Since \( x \) represents time in seconds, \( x \geq 0 \) and the quadratic function will have a restricted domain and it will pass the horizontal line test. Hence, \( h(x) \) a one-to-one function. Answers may vary.
   b. \( h(1.4) = 68.64 \); In 1.4 seconds, the nail gun is 68.64 feet above the ground.
   c. \( h^{-1}(x) = \sqrt{\frac{100 - x}{16}} \); The inverse gives us the time is seconds when the nailer is at a height of \( x \) feet above the ground.
       Answers may vary.
   d. 1.58; When the nail gun is at a height of 60 feet above ground, 1.58 seconds elapsed. Answers may vary.
   e. 2.5 seconds

27. Carly was not given full credit since the function \( f(x) = \frac{2}{x - 6} \) is undefined at \( x = 6 \).
In exercises 1–4, determine whether each of the following functions is one-to-one. If the function is not one-to-one, explain why.

1. \( f(x) = 4\sqrt{x}, x \geq 0 \)
2. \( g(x) = x - x^2 \)
3. \( f = \{(3, -24), (-3, -12), (6, -42), (1, -12)\} \)
4. \( f = \{(1, 9), (9, 27), (10, 28.46), (16, 36)\} \)

In exercises 5–6, determine whether each table represents a one-to-one function. If not, explain why.

5. 

<table>
<thead>
<tr>
<th>Input</th>
<th>-7.2</th>
<th>-2.5</th>
<th>-1</th>
<th>8.58</th>
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</thead>
<tbody>
<tr>
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<td>8.5</td>
<td>-53.25</td>
</tr>
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</table>

6.

<table>
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<th>-10</th>
<th>8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>15</td>
<td>11</td>
<td>25</td>
<td>11</td>
</tr>
</tbody>
</table>

In exercises 7–8, determine if the graphs represent functions that have an inverse; explain why or why not.

7. 

8. 

In exercise 9, state the domain and range of the inverse, given the following.

9. \( f = \{(-6, 20), (-4, 14), (1, -1), (9, -25)\} \)

In exercise 10, given the following values do the following.

a. Make a table of values for \( f(x) \) and another table for its inverse, \( f^{-1}(x) \).

b. State the domain and range of each function.

10. \( f(-12) = -68, f(0) = -8, \) and \( f(3) = 7 \)

In exercises 11–17, find the inverse function for each of the following.

11. \( f(x) = -12x + 8 \)
12. \( f(x) = -28x - 6 \)
13. \( f(x) = -\frac{x + 4}{2} \)
14. \( f(x) = -\frac{3}{9x - 4} \)
15. \( f(x) = \frac{3}{4}x - 3 \)
16. \( f(x) = -\frac{6}{5}x - \frac{1}{2} \)
17. \( f(x) = \frac{7x + 5}{7x - 1} \)
In exercises 18–19, the graph of a function \( f(x) \), is shown; graph \( f(x) \) and \( f^{-1}(x) \) along with the line \( y = x \).

18. 19.

In exercises 20–21 the graph of a function is given. Graph its inverse \( f^{-1}(x) \) using the 2 points shown as a guide. Re-label the points and the tick marks accordingly.

20. 21.

In exercises 22–23, find the inverse of each function, if it exists. If the inverse exists, state the domain and range for the original function and the inverse. If the inverse does not exist, explain why using a complete sentence.

22. \( f(x) = \sqrt{x + 5} \)
23. \( f(x) = \sqrt{x - 4} + 3 \)

24. Jamie purchased a new car this week after deciding to give his old car to his son, who just graduated from college. He paid a dealers’ fee and also paid 6.5% of the purchase price for taxes. The equation for the total cost of the car, \( f(x) \) is given by \( f(x) = 22500 + 0.065x \), where \( x \) is the purchase price.
   a. Find \( f(21000) \) and interpret its meaning.
   b. Find the inverse function.
   c. Find \( f^{-1}(23865) \) and interpret its meaning.
   d. Using your answers from parts (a) and (b), determine how much he paid for the dealer fees and for taxes.

25. Mickey is a race enthusiast who decides to buy a used top fuel dragster. The equation for the total cost of his dragster is given by \( f(x) = 23500 + 0.07x \), where \( x \) is the purchase price. The cost of the used dragster include title fees and also 7% of the purchase price for taxes.
   Little Facts: Gasoline powered Pro-Stock cars produce approximately 1200 horsepower, whereas a Top Fuel dragster produces around 7,000 horsepower that can cover a quarter-mile in just 4.4 seconds, at a speed of over 300 miles per hour.
   Source: www.nhra.net/streetlegal/funfacts.html
   a. Find \( f(21900) \) and interpret its meaning.
   b. Find the inverse function.
   c. Find \( f^{-1}(25033) \) and interpret its meaning.
   d. Using your answers from parts (a) and (b), determine how much Mickey paid in taxes and how much he paid in title fees.

26. An object is dropped from a height of 165 feet. Its height in feet, \( h \), after \( t \) seconds is modeled by the function \( h(t) = -16t^2 + 165 \) for \( t \geq 0 \).
   a. Find \( h(2.9) \) to 2 decimal places and interpret your answer.
   b. Find the inverse of the given function.
   c. Find \( h^{-1}(100) \) to 3 decimal places and interpret its meaning.
27. The profit of a company, in thousands, can be approximated by \( P(x) = \sqrt{12.9x + 3.8} \), where \( x \) denotes time in months for \( x \geq 0 \).
   a. Find \( P(8) \) to 2 decimal places and interpret your answer.
   b. Find the inverse of the given function.
   c. \( P^{-1}(19.5) \) to the nearest whole number and interpret its meaning.

28. The cost of a company, in thousands of dollars, can be approximated by \( C(x) = \sqrt{21.6x - 4.7} \), where \( x \) denotes time in months for \( x \geq 0 \).
   a. Find \( C(5) \) to 2 decimal places and interpret your answer.
   b. Find the inverse of the given function.
   c. \( C^{-1}(20) \) to the nearest whole number and interpret its meaning.

29. An employee of an Internet sales ad company receives a salary of $56,000 plus 4.75% commission on sales. The function \( S(x) = 56000 + 0.0475x \) describes the employee’s salary, where \( x \) represents the amount of sales in dollars.
   a. Complete the table of values for \( S(x) \):

   \[
   \begin{array}{c|c|c|c|c}
   x & 22,000 & 35,000 & 76,000 & 125,000 \\
   S(x) & & & & \\
   \end{array}
   \]

   b. Find \( S^{-1}(x) \) symbolically.
   c. Complete the table of values for \( S^{-1}(x) \):

   \[
   \begin{array}{c|c|c|c|c}
   x & 59,610 & 61,000 & 62,400 & 77,900 \\
   S(x) & & & & \\
   \end{array}
   \]

   d. Find and interpret \( S(91560) \) to the nearest whole number.
   e. Find and interpret \( S^{-1}(63000) \) to the nearest whole number.

30. Magda sells water purifiers for homes and business and relies mostly on sales for her income. She makes a base pay of $30,000 and receives 16% commission in sales.
   a. Write a function that models Magda’s total annual income, \( I \), in dollars, in terms of total sales, \( x \), in dollars.
   b. Find \( I(92000) \) and interpret its meaning.
   c. Use your grapher to graph this function. Include your selected window, and label the axes in the context of the problem. Explain why this is a one-to-one function.
   d. Find the inverse function.
   e. Find and interpret \( I^{-1}(44720) \) in terms of the problem.

31. A boys basketball team plans to make sweatbands to raise money for an out of state event. Their fixed cost is $190 for the terry cloth material and their cost to pay someone from a local company to make the sweatbands is $8.00 each.
   a. Write a function that models the team’s cost function \( C(x) \), in terms of the number of \( x \) sweatbands sold.
   b. Find \( C(20) \) and interpret its meaning.
   c. Use your grapher to graph this function. Include your selected window and label the axes in the context of the problem. Explain why this is a one-to-one function.
   d. Find the inverse function.
   e. Find and interpret \( C^{-1}(640) \) in terms of the problem. Round your answer to the nearest whole number.
   f. Find and interpret \( C^{-1}(400) \) in terms of the problem. Round your answer to the nearest whole number.

In exercises 32–33, find each of the following.

a. \((f + g)(x)\)  
b. \((f - g)(x)\)  
c. \((f \cdot g)(x)\)  
d. \((f / g)(x)\)

32. \( f(x) = 2x^2 - 4x \) and \( g(x) = 6x + 2 \)  
33. \( f(x) = 2x - 3 \) and \( g(x) = 4x^2 - 9 \)
In exercises 34–35, find each of the following and state the domain for each composite function.

a. \((f \circ g)(x)\)  
b. \((g \circ f)(x)\)  
c. \((h \circ f)(x)\)  
d. \((f \circ h)(x)\)

34. \(f(x) = -6x - 8, \ g(x) = -3x^2 - 4x,\) and \(h(x) = \frac{-5}{9x + 2}\)

35. \(f(x) = 5x - 6, \ g(x) = -3x^2 + 1,\) and \(h(x) = \sqrt{2x + 11}\)

In exercises 36–37, find each of the following.

a. \((f \circ g)(x)\)  
b. \((g \circ f)(x)\)

36. \(f(x) = x^2 + 5\) and \(g(x) = \sqrt{3 - 8x}\)  
37. \(f(x) = -8x^2 - 20\) and \(g(x) = \sqrt{7x + 9}\)

38. Given \(f(x) = -3x^2 - 7x + 1\) and \(g(x) = -6x + 10\), find the following.

a. \((f + g)(-2)\)  
b. \((f - g)(5)\)  
c. \((f \cdot g)(-1)\)  
d. \(\left(\frac{f}{g}\right)(3)\)

39. Given \(f(x) = \sqrt{2x + 4}\) and \(g(x) = -2x - 12\), find the following.

a. \((f + g)(6)\)  
b. \((f - g)(16)\)  
c. \((f \cdot g)(0)\)  
d. \(\left(\frac{f}{g}\right)(-3)\)

Use your calculator for exercises 40–42 to find the following. Leave your answers in decimal form and round to 2 decimals places where needed.

40. \(f(x) = \frac{13.5x - 7}{2x}\) and \(g(x) = 2x^2 - 7.4x\)

a. \((f + g)\left(\frac{-3}{8}\right)\)  
b. \((f - g)(6.2)\)  
c. \((f \cdot g)\left(\frac{-15}{7}\right)\)  
d. \(\left(\frac{f}{g}\right)(-1.07)\)

41. \(f(x) = -3x^2 + 11x\) and \(g(x) = \sqrt{4x^2 - 5x}\)

a. \((f \circ g)(-3.2)\)  
b. \((g \circ f)(3)\)

42. \(f(x) = \frac{6x - 7x^2}{2x}\) and \(g(x) = |2x - 9|\)

a. \((f \circ g)(-6.5)\)  
b. \((g \circ f)\left(\frac{-7}{3}\right)\)

In exercises 43–44, given the following graphs find the following.

43. a. \((f - g)(0)\)  
b. \(\left(\frac{f}{g}\right)(-8)\)  
c. \((f \circ g)(-2)\)  
d. \((g \circ f)(3)\)  
e. \((f + g)(-4)\)
44. a. \((f + g)(5)\)  
   b. \((f \cdot g)(6)\)  
   c. \((f \circ g)(0)\)  
   d. \((g \circ f)(1)\)  
   e. \((f \circ g)(-3)\)

45. A company manufactures and sells Christmas wooden nutcrackers during the holidays. The revenue in dollars for the company for \(x\) nutcrackers sold is modeled by \(R(x) = x(9 - 0.01x)\). The total cost of producing the nutcrackers involves a fixed cost of $200, plus a variable cost of $4 per unit.

   Little Facts: In the 1800s, nutcrackers in the form of standing soldiers and kings were shown in German regions during the time when the term "Nussknacker" appeared in the German Brothers Grimm dictionary. Wilhelm Fichtner is known as the "father of the nutcracker" due to making the first commercial production using a lathe. Sources: www.nutcrackermuseum.com; www.thechristmashausonline.com

   a. Write the total cost function \(C(x)\), for this problem.
   b. Write the profit function \(P(x)\).
   c. Find and interpret \(P(325)\). Round to 2 decimal places.

46. Louise’s Bird Company makes wooden bird feeders. The revenue for her company can be modeled by \(R(x) = x(22 - 0.01x)\) and the total cost to the company for producing \(x\) number of bird feeders is $350 for fixed costs and a variable cost of $5 to construct each feeder.

   a. Write the total cost function \(C(x)\).
   b. Write the profit function \(P(x)\).
   c. Find and interpret \(P(60)\).

47. Two engineering groups are having a "launch a vegetable" contest. The mechanical engineering group is launching potatoes and the chemical engineering group chooses turnips. The average height in feet of the launched potatoes can be modeled by \(f(t) = -16t^2 + 87t + 4\) and the average height of the launched turnips is approximated by \(g(t) = -16t^2 + 65t + 5\).

   a. Find \((f + g)(t)\) and interpret its meaning.
   b. Find \((f + g)(2.6)\) to 2 decimal places and interpret its meaning.
   c. Find \((f - g)(t)\) and interpret its meaning.
   d. Find \((f - g)(3.2)\) to 1 decimal place and interpret its meaning.

48. Delmar is on the Nelson Mandela Bridge and uses a sling shot to launch a rock with an initial upward velocity of 48 feet per second at a height of 89 feet. The height in feet, \(h\), of Delmar’s rock can be described by \(h_1(t) = -16t^2 + 48t + 89\), where \(t\) is time in seconds. His friend Baruti is on the Bloukrans Bridge and uses a sling shot as well. His rock has an initial upward velocity of 35 feet per second at a height of 708 feet, and the height of his rock is described by \(h_2(t) = -16t^2 + 35t + 708\).

   Little Facts: Located in South Africa, the Bloukrans Bridge became famous in 1997 when the Face Adrenalin bungee company advertised it for being the world’s highest bungee jump. Opened in 1984, the bridge stands at 708 feet high with a 892 foot span. The name Bloukrans means 'Blue Ridge.' Sources: www.faceadrenalin.com; www.highestbridges.com

   a. Find \((h_1 + h_2)(t)\) and interpret its meaning.
   b. Find \((h_1 + h_2)(5)\) and interpret its meaning.
   c. Find \((h_2 + h_1)(t)\) and interpret its meaning.
   d. Find \((h_2 + h_1)(4.1)\) to 1 decimal place and interpret its meaning.
49. A shoe store is having a “Black Friday” sale on all their tennis shoes. The store is offering a rebate of $10 plus a 25% discount. Find the following if the suggested retail price of the tennis shoes is $p$ dollars.

**Little Facts:** Black Friday is the day after Thanksgiving in the United States, and is traditionally one of the busiest retail shopping days of the year. “Black Friday” historically comes from the retailers’ shift to profitability during the holiday season. Back in the days when accounting records were kept by hand, red ink indicated financial loss while black ink indicated profit. Sources: www.blackfridayandcybermonday.com; www.retailindustry.about.com

a. Write a function, $R(p)$, that represents the cost of the tennis shoes after the store’s rebate.

b. Write a function, $D(p)$, that represents the cost of the shoes after the 25% discount.

c. Find and interpret $(R \circ D)(p)$.

d. Find and interpret $(R \circ D)(80)$.

e. Find and interpret $(D \circ R)(p)$.

f. Find and interpret $(D \circ R)(80)$.

g. Which composite function is the better option for the consumer, $R \circ D$ or $D \circ R$? Explain your decision.

In exercises 50–51, determine whether the given functions $f$ and $g$ are inverses of each other.

50. $f(x) = -8x - 4$ and $g(x) = -\frac{x}{8} - 2$

51. $f(x) = \frac{9x - 1}{3}$ and $g(x) = \frac{3x + 1}{9}$

In exercises 52–53, find the following.

a. The inverse of the function $f(x)$.

b. The domain and range for both $f(x)$ and $f^{-1}(x)$.

c. Graph both $f(x)$ and $f^{-1}(x)$; label axes and tick marks.

52. $f(x) = \sqrt{15 + 5x}$ for $x \geq -3$

53. $f(x) = \frac{x + 4}{5}$ for $x \geq -4$

54. The graph of a function, $f(x)$, is shown.

a. Graph $f^{-1}(x)$ along with the line $y = x$. Use the given plotted points as a guide.

b. Find $(f^{-1} \circ f)(6)$ and plot the points.

c. Find $(f \circ f^{-1})(6)$ and plot the points.

55. The graph of a function, $f(x)$, is shown.

a. Graph $f^{-1}(x)$ along with the line $y = x$. Use the given plotted points as a guide.

b. Find $(f^{-1} \circ f)(6)$ and plot the points.

c. Find $(f \circ f^{-1})(6)$ and plot the points.
56. Vladimir and some friends decided to go canoeing during the weekend. They paid a deposit of $45 to rent a canoe and $5.75 per hour. The total cost, \( C(x) \), to rent the canoe can be represented by the function \( C(x) = 45x + 5.75 \), where \( x \) represents the number of hours they canoed.
   a. Find and interpret \( C^{-1}(x) \).
   b. Find and interpret \( C^{-1}(185.75) \).

57. An air conditioning technician charges $65 for a service call, and then charges $55 per hour. The total cost, \( C(x) \), to the customer can be represented by the function \( C(x) = 55x + 65 \), where \( x \) represents the number of hours the technician worked.
   a. Find \( C(2.75) \) to 2 decimal places and interpret.
   b. Find and interpret \( C^{-1}(x) \).
   c. Find \( C^{-1}(133.75) \) to 2 decimal places and interpret.
   d. Verify algebraically that \( C \) and \( C^{-1} \) are inverses of each other.

58. A little boy drops a teddy bear 130 feet out of hotel window to his sister below standing on the lawn. The height in feet, \( h \), of his rock is given by the function \( h(t) = -16t^2 + 130 \), where \( t \) is time in seconds.

**Little Facts:** In 1902, President Theodore (Teddy) Roosevelt was helping settle a border dispute between Mississippi and Louisiana. During his spare time he went bear hunting. Since the bear hunt had been unsuccessful, the guides captured an injured bear and invited Roosevelt to get his shot. Roosevelt refused to fire on a helpless target and ordered that the bear be put down to end its pain. The next day, a newspaper had created a cartoon and a story depicting the president’s sportsmanship, as shown in his refusal to shoot a captive bear. The cartoon inspired Brooklyn candy store owner Rose Mitchom to create a small, jointed stuffed bear which she called “Teddy’s Bear.” Sources: www.teddybearandfriends.com; www.theodororoossevelt.org; www.nationalgeographic.com

   a. Is \( h(t) \) a one-to-one function? Explain your decision.
   b. Find and interpret \( h(1.8) \) to 2 decimal places.
   c. Find and interpret \( h^{-1}(t) \).
   d. Find and interpret \( h^{-1}(56) \) to 2 decimal places.
   e. When will the teddy bear reach the lawn below where his sister is standing? Round your answer to 2 decimal places.

---

**Answer Key**

Chapter 5 Review Exercises

1. One-to-one
2. Not one-to-one; this represents a parabola, which fails the horizontal line test.
3. Not one-to-one; two different inputs produce the same output.
4. One-to-one
5. One-to-one
6. Not one-to-one; two different inputs produce the same output.
7. No, does not pass the horizontal line test.
8. Yes, passes the horizontal line test.
9. \( D = \{20, 14, -1, -25\} \); \( R = \{-6, -4, 1, 9\} \)
10. a. 

<table>
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<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f^{-1}(x) )</th>
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</tr>
<tr>
<td>3</td>
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<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>

   b. \( f(x) \): \( D = \{-12, 0, 3\} \); \( R = \{-68, -8, 7\} \); \( f^{-1}(x) \): \( D = \{-68, -8, 7\} \); \( R = \{-12, 0, 3\} \)

11. \( f^{-1}(x) = \frac{1}{12}x + \frac{2}{3} \)
12. \( f^{-1}(x) = -\frac{1}{28}x - \frac{3}{14} \)
13. \( f^{-1}(x) = -2x - 4 \)
14. \( f^{-1}(x) = -\frac{1}{3x} + \frac{4}{9} \)
15. \( f^{-1}(x) = \frac{4}{3}x + 4 \)
16. \( f^{-1}(x) = -\frac{5}{6}x - \frac{5}{12} \)
17. \( f^{-1}(x) = \frac{x + 5}{7x - 2} \)

18. Finding the inverse would yield the parabola \( y = x^2 - 5 \), where two distinct \( x \)-values would yield the same \( y \)-value; therefore the inverse does not exist, unless the domain is restricted to \([-5, \infty)\).

20. \( f^{-1}(x) = (x - 3)^3 + 4 \); The inverse exists and the graph of \( f(x) \) and \( f^{-1}(x) \) pass the horizontal line test. The domain and range for both functions is \((\infty, \infty)\).

22. Finding the inverse would yield the parabola \( y = x^2 - 5 \), where two distinct \( x \)-values would yield the same \( y \)-value; therefore the inverse does not exist, unless the domain is restricted to \([-5, \infty)\).

23. \( f^{-1}(x) = \frac{x - 3}{2} + 4 \); The inverse exists and the graph of \( f(x) \) and \( f^{-1}(x) \) pass the horizontal line test. The domain and range for both functions is \((\infty, \infty)\).

24. a. 23865; If the purchase price was $21,000, the total cost was $23,865.
   b. \( f^{-1}(x) = \frac{x - 22500}{0.065} \) or \( f^{-1}(x) = \frac{x}{0.065} - 346,154 \)
   c. 21000; If the total cost was $23,865, the purchase price was $21,000.
   d. He paid $1500 in dealer fees and $1365 in taxes.

25. a. 25033; If the purchase price was $21,900, the total cost was $25,033.
   b. \( f^{-1}(x) = \frac{x - 23500}{0.07} \) or \( f^{-1}(x) = \frac{x}{0.07} - 335,714.29 \)
   c. 21900; If the total cost was $25,033, the purchase price was $21,900.
   d. He paid $1533 in taxes and $1600 for title fees.

26. a. 30.44; In 2.9 seconds the height of the object is 30.44 feet above ground.
   b. \( h(t) = \sqrt{165 - t} \)
   c. 2.016; The height of the object is 100 feet above ground in 2.016 seconds.

27. a. 10.34; In 8 months the profit reached 10.34 thousand dollars.
   b. \( P^{-1}(x) = \frac{x^2 - 3.8}{12.9} \)
   c. 29; The profit of the company was 19.5 thousand dollars in the 29th month.

28. a. 10.16; In 5 months the cost reached 10.16 thousand dollars.
   b. \( C^{-1}(x) = \frac{x^2 + 4.7}{21.6} \)
   c. 19; The profit of the company was 20 thousand dollars in the 19th month.
29. a.  
\[ \begin{array}{c|cccc} \hline x & 22,000 & 35,000 & 76,000 & 125,000 \\ \hline S(x) & 57,045 & 57,663 & 59,610 & 61,938 \\ \hline \end{array} \]

b. \[ S^{-1}(x) = \frac{x - 56000}{0.0475} \]

c.  
\[ \begin{array}{c|cccc} \hline x & 59,610 & 61,000 & 62,400 & 77,900 \\ \hline S^{-1}(x) & 76,000 & 105,263 & 134,737 & 461,053 \\ \hline \end{array} \]

d. $60,349; \text{ When the sales amount is } $91,560, \text{ the salary is } $60,349.

e. $147,368; \text{ If the salary is } $63,000, \text{ the sales amount is } $147,368.

30. a. \( I = 30,000 + 0.16x \), where \( x \geq 0 \).

b. $44,720; \text{ When the sales amount is } $92,000, \text{ the salary is } $44,720.

c. \( I \): \text{ Income}  

\[ x: \text{Total sales} \]

\[ [0, 200000, 2000] \text{ by } [30000, 90000, 2000] \]

Answers may vary. This is an increasing linear function, which passes the horizontal line test.

d. \( I^{-1}(x) = \frac{x - 30000}{0.16} \) or \( I^{-1}(x) = 6.25x - 187,500 \).

e. $92,000; \text{ When the salary is } $44,720, \text{ the sales are } $92,000.

31. a. \( C(x) = 190 + 8x \), where \( x \geq 0 \).

b. 350; \text{ The cost of selling 20 sweatbands is } $350.

c. Answers may vary. This is an increasing linear function, which passes the horizontal line test.

Window used: \([0, 120, 20]\) by \([80, 200, 80]\)

\[ \begin{array}{c|c} \hline \text{Sweatbands} & \text{Cost} \\ \hline 20 & 80 \\ 40 & 160 \\ 60 & 240 \\ 80 & 320 \\ 100 & 400 \\ 120 & 480 \\ \hline \end{array} \]

\[ I = \text{Income} \]

\[ 30,000 \text{ by } 0.16 \]

\[ 30000 \text{ by } 0.16 \]

d. \( C^{-1}(x) = \frac{x - 190}{8} \) or \( C^{-1}(x) = 0.125x - 23.75 \).

e. 56; \text{ When the cost is } $640, 56 sweatbands were sold.

f. 26; \text{ When the cost is } $400, 26 sweatbands were sold.

32. a. \( 2x^2 + 2x + 2 \)

b. \( 2x^2 - 10x - 2 \)

c. \( 12x^3 - 20x^2 - 8x \)

d. \( \frac{x^2 - 2x}{3x + 1} \)

33. a. \( 4x^2 + 2x - 12 \)

b. \( -4x^2 + 2x + 6 \)

c. \( 8x^3 - 12x^2 - 18x + 27 \)

d. \( \frac{1}{2x + 3} \)
34. a. $18x^2 + 24x - 8; (-\infty, \infty)$
   b. $-108x^2 - 264x - 160; (-\infty, \infty)$
   c. $\frac{-5}{54x - 70}; \left(-\infty, \frac{-35}{27}\right) \cup \left(-\frac{35}{27}, \infty\right)$ or $x \neq -\frac{35}{27}$
   d. $\frac{30}{9x + 2} - 8$ or $\frac{72x + 14}{9x + 2}; (-\infty, -\frac{2}{9}) \cup \left(-\frac{2}{9}, \infty\right)$ or $x \neq -\frac{2}{9}$
35. a. $-15x^2 - 1; (-\infty, \infty)$
   b. $-75x^2 + 180x - 107; (-\infty, \infty)$
   c. $\sqrt{10x - 1}; \left(-\frac{1}{10}, \infty\right)$ or $x \geq \frac{1}{10}$
   d. $5\sqrt{2x + 11} - 6; \left(-\frac{11}{2}, \infty\right)$ or $x \geq -\frac{11}{2}$
36. a. $\sqrt{-8x^2 - 37}$
   b. $\sqrt{-8x^2 - 37}$
37. a. $-56x - 92$
   b. $\sqrt{-56x^2 - 131}$
38. a. 25
   b. $-89$
   c. 80
   d. $\frac{47}{8}$
39. a. $-20$
   b. 50
   c. $-24$
   d. 0.98
40. a. 19.14
   b. $-10$
   c. $-24$
   d. $-15$
41. a. $-87.96$
   b. 10.68
42. a. $-74$
   b. $10.34$
43. a. $-6$
   b. $10.34$
   c. 0
   d. 6
44. a. $-15$
   b. 50
   c. $\frac{7}{13}$
   d. $-8$
   e. $-15$
45. a. $C(x) = 4x + 200$
   b. $P(x) = -0.01x^2 + 5x - 200$
   c. $\$368.75$; If the company sells 325 nutcrackers, they can expect a profit of $368.75.$
46. a. $C(x) = 5x + 350$
   b. $P(x) = -0.01x^2 + 17x - 350$
   c. $\$634$; If the company sells 60 bird feeders, they can expect a profit of $634.$
47. a. $-32r^2 + 152t + 9.$ The combined average height of the potatoes and turnips.
   b. $187.88$; The combined average height of the potatoes and turnips after 2.6 seconds is 187.88 feet.
   c. $22t - 1$; The difference in the average height of the potatoes and turnips.
   d. $69.4$; The difference in the average height of the potatoes and turnips after 3.2 seconds is 69.4 feet.
48. a. $-32r^2 + 83t + 797;$ The combined height of the rocks after $t$ seconds.
   b. 412; The combined height of the rocks after 5 seconds is 412 feet.
   c. $-13t + 619;$ The difference in the height of the rocks after $t$ seconds.
   d. 565.7; The difference in the height of the rocks after 4.1 seconds is 565.7 feet.
49. a. $R(p) = p - 10$
   b. $D(p) = 0.75p$
   c. $R(0.75p) = 0.75p - 10$; This is the price after the 25% discount followed by the $10 rebate.
   d. 50; The price of a $80$ pair of tennis shoes discounted at 25% then followed by a $10$ rebate will be $50.$
   e. $D(p - 10) = 0.75(p - 10)$; This is the price after the rebate followed by the discount.
   f. 52.5; The price of a $80$ pair of tennis shoes minus the $10$ rebate then discounted at 25% will be $52.50.$
   g. $R \circ D$ is the better offer. Taking the 25% discount off the regular price first results in a greater price reduction.
50. No
51. Yes
52. \( f^{-1}(x) = \frac{x^2}{5} - 3 \)

b. \( f(x): D = [-3, \infty); R = [0, \infty); \)
\( f^{-1}(x): D = [0, \infty); R = [-3, \infty). \)

c. \( \]

53. \( f^{-1}(x) = 3x^2 - 4 \)

b. \( f(x): D = [-4, \infty); R = [0, \infty); \)
\( f^{-1}(x): D = [0, \infty); R = [-4, \infty). \)

c. \( \]

54. a. \( \]

b. 6

c. 6

55. a. \( \]

b. 6

c. 6

56. a. \( C^{-1}(x) = \frac{x - 5.75}{45} \); The inverse gives us the number of hours canoed when the total cost is \( x \) dollars. Answers may vary.

b. \( C^{-1}(185.75) = 4; \) If the total cost is $185.75, Vladimir and his friends canoed for 4 hours. Answers may vary.

57. a. 216.25; The total cost to the customer for 2.75 hours of work is $216.25. Answers may vary.

b. \( C^{-1}(x) = \frac{1}{55}x - \frac{13}{11}; \) The inverse gives the number of hours the technician worked when the total cost was \( x \) dollars.

\( \]

Answers may vary.

c. 1.25; If the total cost was $133.75, then the technician worked for 1.25 hours. Answers may vary.

d. \( C(C^{-1}(x)) = x \) and \( C^{-1}(C(x)) = x \)

58. a. Since \( t \) represents time in seconds, \( t \geq 0 \) and the quadratic function will have a restricted domain and it will pass the horizontal line test. Hence, \( h(t) \) is a one-to-one function. Answers may vary.

b. 78.16; In 1.8 seconds, the teddy bear is 78.16 feet above the ground.

c. \( h^{-1}(t) = \frac{\sqrt{130 - t}}{16} = \frac{\sqrt{130} - t}{4}. \) The inverse gives us the time is seconds when the teddy bear is at height of \( t \) feet.

\( \]

Answers may vary.

d. 2.15; When the teddy bear is 56 feet above the ground, 2.15 seconds have elapsed. Answers may vary.

e. 2.85 seconds